

Short time large deviations of the KPZ equation

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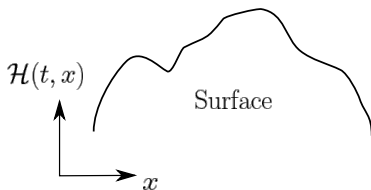
Joint work with Li-Cheng Tsai (Rutgers)

Columbia SPDE seminar

- Introduced by Kardar-Parisi-Zhang (1986)

$$\partial_t \mathcal{H}(t, x) = \frac{1}{2} \partial_{xx} \mathcal{H}(t, x) + \frac{1}{2} (\partial_x \mathcal{H}(t, x))^2 + \xi(t, x)$$

ξ is the space-time white noise.



- Stochastic heat equation (SHE)

$$\partial_t \mathcal{Z}(t, x) = \frac{1}{2} \partial_{xx} \mathcal{Z}(t, x) + \xi(t, x) \mathcal{Z}(t, x)$$

- Well-posed equation
- Solution preserves positivity

Hopf-Cole solution

- KPZ equation

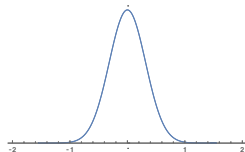
$$\partial_t \mathcal{H}(t, x) = \frac{1}{2} \partial_{xx} \mathcal{H}(t, x) + \frac{1}{2} (\partial_x \mathcal{H}(t, x))^2 + \xi(t, x)$$

- Take $\mathcal{Z} = e^{\mathcal{H}}$, **formally** \mathcal{Z} solves the SHE

$$\partial_t \mathcal{Z}(t, x) = \frac{1}{2} \partial_{xx} \mathcal{Z}(t, x) + \mathcal{Z}(t, x) \xi(t, x)$$

Hopf-Cole solution $\mathcal{H}(t, x) := \log \mathcal{Z}(t, x)$.

- **Narrow wedge initial data:** $\mathcal{Z}(0, x) = \delta_0(x)$, $\mathcal{Z}(t, x) \approx p(t, x)$.

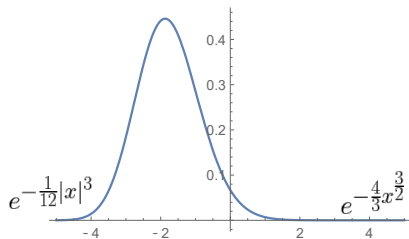


One point fluctuation of KPZ

- [Amir-Corwin-Quastel 11]

(long time) $\frac{\mathcal{H}(2t, 0) + \frac{t}{12}}{t^{\frac{1}{3}}} \implies \text{Tracy Widom GUE} \quad t \rightarrow \infty$

(short time) $\frac{\mathcal{H}(2t, 0) + \log(\sqrt{4\pi t})}{t^{\frac{1}{4}}} \implies \mathcal{N}(0, \sqrt{\frac{\pi}{2}}), \quad t \rightarrow 0$



pdf of Tracy Widom-GUE

- What about Large Deviation Principle (LDP)?

LDP of KPZ, long time regime

- Lower tail LDP [Tsai 18], [Cafasso-Claeys 19]. For $\lambda > 0$,

$$\lim_{t \rightarrow \infty} \frac{1}{t^2} \log \mathbb{P} \left(\mathcal{H}(2t, 0) + \frac{t}{12} < -\lambda t \right) = -\Phi(\lambda)$$

$\Phi(\lambda)$ explicit, $\lim_{\lambda \rightarrow 0^+} \frac{\Phi(\lambda)}{\lambda^3} = \frac{1}{12}$, $\lim_{\lambda \rightarrow \infty} \frac{\Phi(\lambda)}{\lambda^{\frac{5}{2}}} = \frac{4}{15\pi}$

- For finite large t , a tail bound which identifies this crossover was established earlier in [Corwin-Ghosal 18]
- Upper tail LDP [Das-Tsai 19]
- The $\frac{5}{2}$ -power law was first predicted in [Kolokolov-Korshunov 07] for short time regime
- What is the LDP in the short time regime?

Main Result

Theorem (L., Tsai 20)

There exist ϕ_+ , ϕ_- s.t. for any $\lambda > 0$,

$$\lim_{t \rightarrow 0} \sqrt{t} \log \mathbb{P}(\mathcal{H}(2t, 0) + \log(\sqrt{4\pi t}) > \lambda) = -\phi_+(\lambda)$$

$$\lim_{t \rightarrow 0} \sqrt{t} \log \mathbb{P}(\mathcal{H}(2t, 0) + \log(\sqrt{4\pi t}) < -\lambda) = -\phi_-(\lambda)$$

In addition,

$$\lim_{\lambda \rightarrow 0} \frac{\phi_-(\lambda)}{\lambda^2} = \frac{1}{\sqrt{2\pi}}, \quad \lim_{\lambda \rightarrow \infty} \frac{\phi_-(\lambda)}{\lambda^{\frac{5}{2}}} = \frac{4}{15\pi}$$

Conjecture

We have $\lim_{\lambda \rightarrow \infty} \frac{\phi_+(\lambda)}{\lambda^{\frac{3}{2}}} = \frac{4}{3}$.

LDP of KPZ, short time regime

- LDP of KPZ in short time has been studied in the physics literature [Kolokolov-Korshunov 07], [Le Doussal-Majumdar-Schehr-Rosso 16], [Kamenev-Meerson-Sasarov 16]
- We will first explain the heuristic named **weak noise theory** from [Kamenev-Meerson-Sasarov 16]. This heuristic provides **a nice input** for our rigorous proof of the $\frac{5}{2}$ power law.

- Consider scaling $\mathcal{H}_\epsilon(t, x) = \mathcal{H}(\epsilon^2 t, \epsilon x) + \log \epsilon$. Then

$$\partial_t \mathcal{H}_\epsilon = \frac{1}{2} \partial_{xx} \mathcal{H}_\epsilon + \frac{1}{2} (\partial_x \mathcal{H}_\epsilon)^2 + \sqrt{\epsilon} \xi$$

- Study the functional LDP of $\mathcal{H}_\epsilon(\cdot, \cdot)$ on $[0, 2] \times \mathbb{R}$ as $\epsilon \rightarrow 0$, push the functional LDP forward to $\mathcal{H}_\epsilon(2, 0) + \log(\sqrt{4\pi})$.
- Solve a **Hamilton variation problem**.
- **Fact:** The LDP

$$\lim_{\epsilon \rightarrow 0} \epsilon \log \mathbb{P}(\mathcal{H}_\epsilon(2, 0) + \log(\sqrt{4\pi}) > \lambda) = -\phi_+(\lambda)$$

$$\lim_{\epsilon \rightarrow 0} \epsilon \log \mathbb{P}(\mathcal{H}_\epsilon(2, 0) + \log(\sqrt{4\pi}) < -\lambda) = -\phi_-(-\lambda)$$

is equivalent to the LDP of $\mathcal{H}(2t, 0) + \log(\sqrt{4\pi t})$ as $t \rightarrow 0$.

- For $\rho \in L^2([0, 2] \times \mathbb{R})$,

$$-\epsilon \log \mathbb{P}(\sqrt{\epsilon} \xi \approx \rho) \approx \frac{1}{2} \|\rho\|^2, \quad \epsilon \rightarrow 0$$

Note that

$$\partial_t \mathcal{H}_\epsilon - \frac{1}{2} \partial_{xx} \mathcal{H}_\epsilon - \frac{1}{2} (\partial_x \mathcal{H}_\epsilon)^2 = \sqrt{\epsilon} \xi$$

This implies that

$$-\epsilon \log \mathbb{P}(\mathcal{H}_\epsilon \approx h) \approx \frac{1}{2} \left\| \partial_t h - \frac{1}{2} \partial_{xx} h - \frac{1}{2} (\partial_x h)^2 \right\|^2$$

\downarrow
 ρ

- Let $\rho = \partial_t h - \frac{1}{2}\partial_{xx}h - \frac{1}{2}(\partial_x h)^2$, then

$$-\epsilon \log \mathbb{P}(\mathcal{H}_\epsilon \approx h) \approx \frac{1}{2}\|\rho\|^2, \quad \epsilon \rightarrow 0.$$

- We want to find $-\epsilon \log \mathbb{P}(\mathcal{H}_\epsilon(2, 0) \approx -\lambda) \approx -\phi_-(\lambda)$ as $\epsilon \rightarrow 0$
- Given the boundary $h(2, 0) = -\lambda$, minimize

$$F(h) := \frac{1}{2} \int_0^2 \int (\partial_t h - \frac{1}{2}\partial_{xx}h - \frac{1}{2}(\partial_x h)^2)^2 dt dx.$$

- By calculus of variation, we get

$$\partial_t \rho + \frac{1}{2}\partial_{xx}\rho = \partial_x(\rho \partial_x h)$$

$$\rho = \partial_t h - \frac{1}{2}\partial_{xx}h - \frac{1}{2}(\partial_x h)^2$$

- We want the LDP of $\mathcal{H}_\epsilon(2, 0) \approx -\lambda$ as $\lambda \rightarrow \infty$.
 - Scale $h(t, x) \rightarrow \lambda h(t, \lambda^{-\frac{1}{2}}x), \rho \rightarrow \lambda \rho(t, \lambda^{-\frac{1}{2}}x)$
- Under this scaling,

$$\partial_t h - \frac{1}{2\lambda} \partial_{xx} h - \frac{1}{2} (\partial_x h)^2 = \rho$$
$$\partial_t \rho + \frac{1}{2\lambda} \partial_{xx} \rho = \partial_x (\rho \partial_x h)$$

Physics heuristic

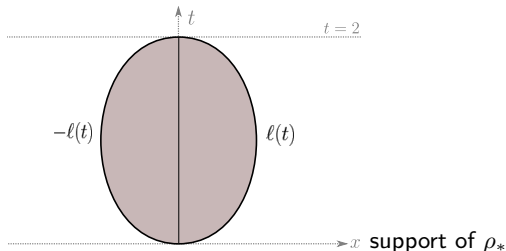
- Remove the viscosity as $\lambda \rightarrow \infty$, we have

$$\partial_t h - \frac{1}{2}(\partial_x h)^2 = \rho$$

$$\partial_t \rho = \partial_x (\rho \partial_x h)$$

- Solve the PDEs, one gets

$$\rho_*(t, x) = -\frac{1}{2\pi\ell(t)} \left(1 - \frac{x^2}{\ell(t)^2}\right) \mathbf{1}_{\{|x| \leq \ell(t)\}}, \quad \frac{1}{2} \|\rho_*\|^2 = \frac{4}{15\pi}$$



- This suggests that $\phi_-(\lambda) \approx \frac{4}{15\pi} \lambda^{\frac{5}{2}}$, $\lambda \rightarrow \infty$.

Major problems and our approach

- Major problems of physics heuristic:
 - Why one can remove the **viscosity**?
 - **uniqueness** of the solution
- Our approach:
 - **Work at the level of SHE**
 - We first prove a functional LDP of the SHE \mathcal{Z}_ϵ . Then we push the functional LDP forward to $\mathcal{H}_\epsilon(2, 0)$. The rate function has a **different** variational form (in terms of Feynman Kac formula).
 - We are able to apply **probabilistic argument** to extract the asymptotic.

- Consider $\mathcal{Z}_\epsilon = e^{\mathcal{H}_\epsilon}$, then

$$\partial_t \mathcal{Z}_\epsilon = \frac{1}{2} \partial_{xx} \mathcal{Z}_\epsilon + \sqrt{\epsilon} \xi \mathcal{Z}_\epsilon, \quad \mathcal{Z}_\epsilon(0, \cdot) = \delta_0(\cdot)$$

- Feynman-Kac formula

$$\frac{\mathcal{Z}_\epsilon(t, x)}{p(t, x)} = \mathbb{E}_{0 \rightarrow x} \left[\exp \left(\int_0^t \sqrt{\epsilon} \xi(s, B_b(s)) ds \right) \right]$$

- **Formal derivation** of functional LDP of $\frac{\mathcal{Z}_\epsilon(\cdot, \cdot)}{p(\cdot, \cdot)} \in C([0, 2] \times \mathbb{R})$: Consider

$$\mathcal{J}(\rho) = \mathbb{E}_{0 \rightarrow x} \left[\exp \left(\int_0^t \rho(s, B_b(s)) ds \right) \right]$$

One should expect $\mathcal{J}(\sqrt{\epsilon} \xi) = \frac{\mathcal{Z}_\epsilon(\cdot, \cdot)}{p(\cdot, \cdot)}$ satisfies a LDP with speed ϵ^{-1} and rate function

$$I(f) = \inf \left\{ \frac{1}{2} \|\rho\|^2 : \rho \in L^2([0, 2] \times \mathbb{R}), \mathcal{J}(\rho) = f \right\}$$

- Rigorous proof of the LDP: Chao series expansion

$$\frac{\mathcal{Z}_\epsilon(t, x)}{p(t, x)} = \sum_{k=0}^{\infty} \mathcal{I}_k(\sqrt{\epsilon}\xi)(t, x)$$

where

$$\mathcal{I}_k(\sqrt{\epsilon}\xi)(t, x) = \int P_k(t, x, t_1, x_1, \dots, t_k, x_k) \prod_{i=1}^k \sqrt{\epsilon}\xi(t_i, x_i)$$

- [Hairer-Weber 15] LDP of $\sum_{k=0}^N \mathcal{I}_k(\sqrt{\epsilon}\xi)(\cdot, \cdot)$ with speed ϵ^{-1} has rate

$$I(f) = \inf \left\{ \frac{1}{2} \|\rho\|^2 : \rho \in L^2([0, 2] \times \mathbb{R}), \sum_{k=0}^N \mathcal{I}_k(\rho) = f \right\}$$

- Exponential approximation

- Hypercontractive inequality \Rightarrow One point tail estimate of $\mathcal{I}_k(\sqrt{\epsilon}\xi)(t, x)$
- Kolmogorov type argument \Rightarrow Functional tail estimate of $\mathcal{I}_k(\sqrt{\epsilon}\xi)(\cdot, \cdot)$

- Functional LDP of $\frac{\mathcal{Z}_\epsilon(\cdot, \cdot)}{p(\cdot, \cdot)} = \sum_{k=0}^{\infty} \mathcal{I}_k(\sqrt{\epsilon}\xi)(\cdot, \cdot)$

$$I(f) = \inf \left\{ \frac{1}{2} \|\rho\|^2 : \rho \in L^2([0, 2] \times \mathbb{R}), \sum_{k=0}^{\infty} \mathcal{I}_k(\rho) = f \right\}$$

- Feynman-Kac

$$\sum_{k=0}^{\infty} \mathcal{I}_k(\rho)(t, x) = \mathbb{E}_{0 \rightarrow x} \left[\exp \left(\int_0^t \rho(s, B_b(s)) ds \right) \right]$$

Theorem (L. Tsai 20)

$\mathcal{Z}_\epsilon(\cdot, \cdot)/p(\cdot, \cdot)$ satisfies a LDP with speed ϵ^{-1} and a good rate function

$$I(f) = \inf \left\{ \frac{1}{2} \|\rho\|^2 : \mathcal{J}(\rho) = f, \rho \in L^2([0, 2] \times \mathbb{R}) \right\}$$

$$\mathcal{J}(\rho)(t, x) = \mathbb{E}_{0 \rightarrow x} \left[\exp \left(\int_0^t \rho(s, B_b(s)) ds \right) \right]$$

We push the functional LDP forward to one point $\mathcal{H}_\epsilon(2, 0) + \log \sqrt{4\pi}$ as $\epsilon \rightarrow 0$

Corollary (L. Tsai 20)

Let $\mathcal{S}(\rho) = (\log \mathcal{J}(\rho))(2, 0)$. We have for $\lambda > 0$,

$$\phi_-(\lambda) = \inf \left\{ \frac{1}{2} \|\rho\|^2 : \mathcal{S}(\rho) \leq -\lambda \right\}$$

$$\phi_+(\lambda) = \inf \left\{ \frac{1}{2} \|\rho\|^2 : \mathcal{S}(\rho) \geq \lambda \right\}$$

Asymptotics of the rate function

- Want to prove $\lim_{\lambda \rightarrow \infty} \frac{\phi_-(\lambda)}{\lambda^{\frac{5}{2}}} = \frac{4}{15\pi}$
- Scale $\rho \rightarrow \lambda\rho(t, \lambda^{-\frac{1}{2}}x)$, then

$$\phi_-(\lambda) = \lambda^{\frac{5}{2}} \inf \left\{ \frac{1}{2} \|\rho\|^2 : \mathcal{S}_\lambda(\rho) \leq -1 \right\}$$

where $\mathcal{S}_\lambda(\rho) = \lambda^{-1} \log \mathbb{E} \left[\exp \left(\lambda \int_0^2 \rho(s, \lambda^{-\frac{1}{2}} B_b(s)) ds \right) \right]$

- Only need to show

$$\liminf_{\lambda \rightarrow \infty} \inf \left\{ \frac{1}{2} \|\rho\|^2 : \mathcal{S}_\lambda(\rho) \leq -1 \right\} = \frac{4}{15\pi}$$

- We have the input from the physics heuristic

$$\rho_*(t, x) = -\frac{1}{2\pi\ell(t)} \left(1 - \frac{x^2}{\ell(t)^2}\right) \mathbf{1}_{\{|x| \leq \ell(t)\}}, \quad \frac{1}{2} \|\rho_*\|^2 = \frac{4}{15\pi}$$

- Step 1: $\lim_{\lambda \rightarrow \infty} \mathcal{S}_\lambda(\rho_*) = -1$
- Step 2: As $\lambda \rightarrow \infty$, prove ρ_* is the minimizer.

Step 1: Varadhan's lemma

- By Varadhan's lemma

$$\begin{aligned}\lim_{\lambda \rightarrow \infty} \mathcal{S}_\lambda(\rho_*) &= \lim_{\lambda \rightarrow \infty} \lambda^{-1} \log \mathbb{E}_{0 \rightarrow 0} \left[\exp \left(\lambda \int_0^2 \rho_*(s, \lambda^{-\frac{1}{2}} B_b(s)) ds \right) \right] \\ &= \sup_{x(0)=x(2)=0} \left(\int_0^2 \rho_*(s, x(s)) - \frac{1}{2} \dot{x}(s)^2 ds \right)\end{aligned}$$

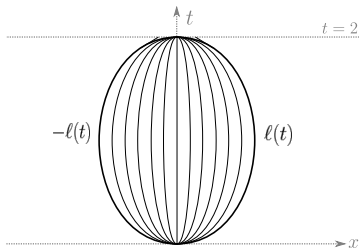
- We solve the Euler-Lagrange Equation

$$\ddot{x}(s) + \partial_x \rho_*(s, x(s)) = 0$$

Geodesic (solution) is not unique.

$$x(t) = \alpha \ell(t), \alpha \in [-1, 1]$$

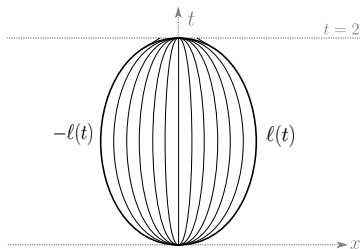
$$\int_0^2 \rho_*(s, x(s)) - \frac{1}{2} \dot{x}(s)^2 ds = -1$$



Step 2: Girsanov's theorem

- We consider $\rho \in L^2([0, 2] \times \mathbb{R})$ such that
$$\mathcal{S}_\lambda(\rho) = \lambda^{-1} \log \mathbb{E} \left[\exp \left(\lambda \int_0^2 \rho(s, \lambda^{-\frac{1}{2}} B_b(s)) ds \right) \right] \leq -1$$
- Jensen's inequality $\mathbb{E} \left[\int_0^2 \rho(s, \lambda^{-\frac{1}{2}} B_b(s)) ds \right] \leq -1$
- Girsanov's theorem $B_b(s) \rightarrow B_b(s) + \lambda^{\frac{1}{2}} x(s)$ and Jensen's inequality,

$$\mathbb{E} \left[\int_0^2 \rho(s, \lambda^{-\frac{1}{2}} B_b(s) + x(s)) ds \right] - \frac{1}{2} \int_0^2 \dot{x}(s)^2 ds \leq -1$$



- Take $\lambda = \infty$ and $x(s) = \alpha l(s)$, we get for $\alpha \in [-1, 1]$,

$$\int_0^2 \rho(s, \alpha l(s)) ds - \frac{1}{2} \alpha^2 \int_0^2 \dot{l}(s)^2 ds \leq -1.$$

- For any $\alpha \in [-1, 1]$,

$$\int_0^2 \rho(s, \alpha \ell(s)) ds - \rho_*(s, \alpha \ell(s)) ds \leq 0.$$

- Multiplying above by $(1 - \alpha^2)_+$ and integrate against α ,

$$\int \int_0^2 (1 - \alpha^2)_+ (\rho(s, \alpha \ell(s)) - \rho_*(\alpha \ell(s))) ds d\alpha \leq 0$$

- Change of variable $\alpha \ell(s) \rightarrow x$, we get

$$\int \int_0^2 \underbrace{\frac{1}{\ell(s)} \left(1 - \frac{x^2}{\ell(s)}\right)}_{-2\pi \rho_*(s, x)} (\rho(s, x) - \rho_*(s, x)) ds dx \leq 0$$

- This implies that $\|\rho\| \geq \|\rho_*\|$.

- We prove a functional LDP of the SHE equation under weak noise scaling. We push this LDP forward to the one point LDP of the KPZ equation.
- By probabilistic arguments, we show a crossover in the lower tail rate function from exponent 2 to $\frac{5}{2}$.
- The proof also works for the KPZ with flat initial data predicted in [Kolokolov-Korshunov 09], [Meerson-Katzav-Vilenkin 16].
- $\frac{5}{2}$ power law also appears in TASEP [Derrida-Lebowitz 98].
 - Is this power law “universal”?

Thank you