Lyapunov exponents of the SHE with general initial data

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Outline

1. Background
2. Results and heuristics
3. Ingredients of the proof.

Stochastic heat equation

The stochastic heat equation (SHE)

$$\partial_t \mathcal{Z}(t,x) = \frac{1}{2}\partial_{xx}\mathcal{Z}(t,x) + \mathcal{Z}(t,x)\xi(t,x).$$

$\xi(t,x)$ is the space-time white noise.

Hopf-Cole transform $\mathcal{Z}(t,x) = e^{\mathcal{H}(t,x)}$, $\mathcal{H}(t,x)$ is the solution to the KPZ equation (Kardar-Parisi-Zhang 86')

$$\partial_t \mathcal{H}(t,x) = \frac{1}{2}\partial_{xx}\mathcal{H}(t,x) + \frac{1}{2}(\partial_x \mathcal{H}(t,x))^2 + \xi(t,x).$$

Lyapunov exponent

Two types of initial datas: 1. Narrow wedge $\mathcal{Z}^{\textbf{nw}}(0,x) = \delta_0(x)$. 2. General initial data: $\mathcal{Z}^f(0,x) = e^{f(x)}$, $f : \mathbb{R} \to \mathbb{R}$ measurable (can be random).

Definition

Define the p-th Lyapunov exponent to be (if the limit exists)

$$\gamma_p := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\big[\mathcal{Z}(t,0)^p\big]$$

Let $\gamma_p^{\textbf{nw}}$ and γ_p^f be the Lyapunov exponent of SHE with narrow wedge or general initial data $e^{f(x)}$.

Lyapunov exponents and LDP

Discovered by [Das-Tsai 19] for the narrow wedge case.

Lyapunov exponents $\gamma_p = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\big[\mathcal{Z}(t,0)^p\big]$

If γ_p exists for every $p > 0$, we have LDP

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(\mathcal{H}(t,0) > st\Big) = -\sup_{p>0} \left(sp - \gamma_p\right)$$

for $s > \eta$, $\eta = \lim_{p\to 0} \left(\frac{d}{dp}\gamma_p\right)$.

Proof of upper bound: Markov inequality

$\mathbb{P}\big(\mathcal{H}(t,0) > st\big) \le e^{-pst}\mathbb{E}\big[e^{p\mathcal{H}(t,0)}\big] = e^{-pst}\mathbb{E}\big[\mathcal{Z}(t,0)^p\big]$

$\frac{1}{t} \log \mathbb{P}\big(\mathcal{H}(t,0) > st\big) \le -ps + \frac{1}{t} \log \mathbb{E}\big[\mathcal{Z}(t,0)^p\big] \to -ps + \gamma_p$

Lyapunov exponent of SHE

- Computation goes back to (Bertini-Cancrini 95')
- (Chen 15') $\gamma_p^f = \frac{p^3-p}{24}$ for integer p, bounded deterministic f.
- (Corwin-Ghosal 18') identifies $\gamma_p^{\mathbf{nw}} = \frac{p^3-p}{24}$ for integer p.
- (Das-Tsai 19') shows the above equality holds for all $p > 0$. As a result, they obtain the upper tail LDP for the KPZ equation starting from narrow wedge data.

Some Motivations

(Das-Tsai 19') computes $\gamma_p^{\textbf{nw}} = \frac{p^3-p}{24}$ using the exact formula of $\mathcal{Z}^{\textbf{nw}}(t,0)$ in (Amir-Corwin-Quastel 11').

Want to compute Lyapunov exponent of $\mathcal{Z}^{\textbf{BM}}$ with $e^{f(x)} = e^{B(x)}$, $B(x)$ is a two sided Brownian motion. Exact formula exists in (Borodin-Corwin-Ferrari-Veto 14') for $\mathcal{Z}^{\textbf{BM}}(t,0)$, but too complicated to play with.

We manage to compute it without using exact formula.

A special case of our result

Theorem (Ghosal, L. 20')

Take f to be a two sided Brownian motion. Then

1. *For every $p > 0$, $\gamma_p^f = \frac{p^3}{6} - \frac{p}{24}$*
2. *For $s > 0$,*
$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(\mathcal{H}^f(t,0) + \frac{t}{24} > st\Big) = -\frac{2\sqrt{2}}{3} s^{\frac{3}{2}}.$$

In fact, we can obtain Lyapunov exponents and LDP for a wide class of f.

Heuristics

Lemma (Corwin, Hammond 16')

$$\mathcal{Z}^f(t,0) \stackrel{d}{=} \int_{-\infty}^{\infty} \mathcal{Z}^{\mathbf{nw}}(t,x) e^{f(x)} dx$$

where $\mathcal{Z}^{\mathbf{nw}}$ and f are independent.

We want to compute

$$\mathbb{E}\big[\mathcal{Z}^f(t,0)^p\big] = \mathbb{E}\Big[\Big(\int_{-\infty}^{\infty} \mathcal{Z}^{\mathbf{nw}}(t,x) e^{f(x)} dx\Big)^p\Big]$$

The following should hold heuristically

$$\mathbb{E}\big[\mathcal{Z}^f(t,0)^p\big] \approx \mathbb{E}\Big[\int_{-\infty}^{\infty} \mathcal{Z}^{\mathbf{nw}}(t,x)^p e^{pf(x)} dx\Big]$$

Heuristics

Once this is established, using Fubini,

$$\mathbb{E}\big[\mathcal{Z}^f(t,0)^p\big] \approx \mathbb{E}\Big[\int_{-\infty}^{\infty} \mathcal{Z}^{\mathbf{nw}}(t,x)^p e^{pf(x)} dx\Big]$$

$$= \int_{-\infty}^{\infty} \mathbb{E}\big[\mathcal{Z}^{\mathbf{nw}}(t,x)^p\big]\mathbb{E}\big[e^{pf(x)}\big] dx$$

Proposition (Amir-Corwin-Quastel 11')

Fix $t > 0$, $\{\mathcal{H}^{\mathbf{nw}}(t,x) + \frac{x^2}{2t}, x \in \mathbb{R}\}$ is stationary in x.

$$\mathbb{E}\big[\mathcal{Z}^{\mathbf{nw}}(t,x)^p\big] = \mathbb{E}\big[\mathcal{Z}^{\mathbf{nw}}(t,0)^p\big] e^{-\frac{px^2}{2t}}$$

$$\mathbb{E}\big[\mathcal{Z}^f(t,0)^p\big] \approx \mathbb{E}\big[\mathcal{Z}^{\mathbf{nw}}(t,0)^p\big] \int \mathbb{E}\big[e^{pf(x)}\big] e^{\frac{-px^2}{2t}} dx$$

Heuristics

$$\log \mathbb{E}\big[\mathcal{Z}^f(t,0)^p\big] \approx \log \mathbb{E}\big[\mathcal{Z}^{\textbf{nw}}(t,0)^p\big] + \log \int \mathbb{E}\big[e^{pf(x)}\big] e^{\frac{-px^2}{2t}} dx$$

Asymptotic of $\mathbb{E}\big[\mathcal{Z}^{\textbf{nw}}(t,0)^p\big]$ is given through

$$\gamma_p^{\textbf{nw}} = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\big[\mathcal{Z}^{\textbf{nw}}(t,0)^p\big] = \frac{p^3-p}{24}.$$

f is a two sided Brownian motion.

$$\lim_{t\to\infty} \frac{1}{t} \log \int \mathbb{E}\big[e^{pf(x)}\big] e^{-\frac{px^2}{2t}} dx = \lim_{t\to\infty} \frac{1}{t} \log \int e^{\frac{p^2|x|}{2} - \frac{px^2}{2t}} dx = \frac{p^3}{8}.$$

So $\gamma_p^{\textbf{BM}} = \frac{p^3-p}{24} + \frac{p^3}{8} = \frac{p^3}{6} - \frac{p}{24}$.

Condition on initial data

Expect: $\lim_{t\to\infty} \frac{1}{t} \log \left(\int \mathbb{E}\left[e^{pf(x)}\right] e^{-\frac{px^2}{2t}} dx \right) = g(p)$.

Consider f satisfying the conditions: there exists $g : \mathbb{R}_{>0} \to \mathbb{R}$ s.t. for all $p > 0$,

1. $$\lim_{t\to\infty} \frac{1}{t} \sup_{x\in\mathbb{R}} (\log \mathbb{E}\left[e^{pf(x)}\right] - \frac{px^2}{2t}) = g(p)$$
2. $$\liminf_{\epsilon\to 0} \lim_{t\to\infty} \frac{1}{t} \log \int e^{-\frac{p(1-\epsilon)x^2}{2t}} \mathbb{E}\left[e^{p(1+\epsilon)f(x)}\right] dx \leq g(p)$$
3. There exists $C, \kappa, \delta > 0$ s.t. for all $s > 0$ and x, y s.t. $|x - y| \leq 1$
$$\mathbb{P}\Big(|f(x) - f(y)| > s\Big) \leq C \exp(-\kappa s^{1+\delta})$$

Main Result

Theorem (Ghosal, L. 20')

If there exists g s.t. the conditions are true. Then

1. $\gamma_p^f = \frac{p^3 - p}{24} + g(p)$.
2. *If additionally $g \in C^1(\mathbb{R}_{>0})$, then*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(\mathcal{H}^f(t,0) + \frac{t}{24} > st\Big) = -I(s), \quad s > \zeta.$$

$\zeta := \lim_{p\to 0} g'(p)$ *and* $I(s) = \sup_{s>0} \big(sp - g(p) - \frac{p^3}{24}\big)$.

Main Result

Example 1. f deterministic and satisfy $|f(x)| \leq C(1+|x|^{\delta})$, $\delta < 1$, and the regularity condition. Under this,

$$1).\ \lim_{t\to\infty} \frac{1}{t} \sup_{x\in\mathbb{R}} \big(pf(x) - \frac{px^2}{2t}\big) = 0,$$

$$2).\ \liminf_{\epsilon\to 0} \lim_{t\to\infty} \frac{1}{t} \log\left(\int e^{-\frac{p(1-\epsilon)x^2}{2t} + p(1+\epsilon)f(x)} dx\right) = 0.$$

Therefore, $g = 0$ and $\gamma_p^f = \frac{p^3-p}{24}$.

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(\mathcal{H}^f(t,0) + \frac{t}{24} > ts\Big) = -\frac{4\sqrt{2}}{3} s^{\frac{3}{2}}.$$

Main Result

Example 2. $f(x) = B_1(x) + a_1 x$ if $x > 0$, $f(x) = B_2(x) - a_2 x$ if $x < 0$. B_1, B_2 independent Brownian motion.

$$\log \mathbb{E}\left[e^{pf(x)}\right] = \begin{cases} (pa_1 + \frac{p^2}{2})x & x > 0 \\ (-pa_2 + \frac{p^2}{2})x & x < 0. \end{cases}$$

$g(p) = \lim_{t\to\infty} \frac{1}{t} \sup_{x\in\mathbb{R}} \left(\log \mathbb{E}\left[e^{pf(x)}\right] - \frac{px^2}{2t} \right) = \frac{p}{2}\left((\frac{p}{2} + a)_+ \right)^2$ where $a = \max(a_1, a_2)$.

$$\liminf_{\epsilon\to 0} \lim_{t\to\infty} \frac{1}{t} \log \left(\int e^{-\frac{p(1-\epsilon)x^2}{2t}} \mathbb{E}\left[e^{(p+\epsilon)f(x)}\right] dx \right) = g(p).$$

In addition, f satisfies the regularity condition.

Main Result

Corollary (Ghosal, L.20)

When f is a two side BM with drift a_1, a_2, $a = \max(a_1, a_2)$.

1. $\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[\mathcal{Z}^f(t,0)^p\right] = \frac{p^3-p}{24} + \frac{p}{2}\left(\left(\frac{p}{2}+a\right)_+\right)^2.$
2. *If* $a \geq 0$, *for* $s > \frac{a^2}{2}$,
   $\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\left(\mathcal{H}^f(t,0) + \frac{t}{24} > st\right) = -\frac{2\sqrt{2}}{3} s^{\frac{3}{2}} + sa - \frac{a^3}{6}$
   If $a \leq 0$, *then*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\left(\mathcal{H}^f(t,0) + \frac{t}{24} > st\right)$$
$$\begin{cases} -\frac{4\sqrt{2}}{3} s^{\frac{3}{2}}, & s \in (0, \frac{a^2}{2}) \\ -\frac{2\sqrt{2}}{3} s^{\frac{3}{2}} + sa - \frac{a^3}{6} & s \geq \frac{a^2}{2}. \end{cases}$$

LDP rate function and right tail

Remark

Consider the limit and upper tail large deviation of $t^{-\frac{1}{3}}(\mathcal{H}^f(2t,0)+\frac{t}{12})$ as $t \to \infty$ for various initial conditions, we see that

Initial condition	*Limit law and right tail*	*Upper LDP rate*
Narrow wedge	$F_{GUE}, -\frac{4}{3}s^{\frac{3}{2}}$	$-\frac{4}{3}s^{\frac{3}{2}}$
Flat	$F_{GOE}(4^{\frac{1}{3}}\cdot), -\frac{4}{3}s^{\frac{3}{2}}$	$-\frac{4}{3}s^{\frac{3}{2}}$
BM	$F_0, -\frac{2}{3}s^{\frac{3}{2}}$	$-\frac{2}{3}s^{\frac{3}{2}}$

The LDP rate function coincides the right tail of the limit law.

Proof idea

Want to make sense

$$\mathbb{E}\Big[\Big(\int_{-\infty}^{\infty} \mathcal{Z}^{\mathbf{nw}}(t,x)e^{f(x)}dx\Big)^p\Big] \approx \mathbb{E}\Big[\int_{-\infty}^{\infty} \mathcal{Z}^{\mathbf{nw}}(t,x)^p e^{pf(x)}dx\Big]$$

(lower bound) $\liminf_{t\to\infty} \frac{1}{t}\log \mathbb{E}\big[\mathcal{Z}^f(t,0)^p\big] \geq g(p) + \gamma_p^{\mathbf{nw}}$.

(upper bound) $\limsup_{t\to\infty} \frac{1}{t}\log \mathbb{E}\big[\mathcal{Z}^f(t,0)^p\big] \leq g(p) + \gamma_p^{\mathbf{nw}}$

Need respectively

1. $\lim_{t\to\infty} \frac{1}{t}\sup_{x\in\mathbb{R}}\big(\log \mathbb{E}\big[e^{pf(x)}\big] - \frac{px^2}{2t}\big) = g(p)$
2. $\liminf_{\epsilon\to 0} \lim_{t\to\infty} \frac{1}{t}\log \int e^{-\frac{p(1-\epsilon)x^2}{2t}} \mathbb{E}\big[e^{p(1+\epsilon)f(x)}\big]dx \leq g(p)$

Upper bound, $p > 1$

By Hölder inequality, $q = \frac{p}{p-1}$,

$$\mathcal{Z}^f(t,0) = \int e^{-\frac{\epsilon x^2}{2t}} \Big(e^{\frac{\epsilon x^2}{2t}} \mathcal{Z}^{\mathbf{nw}}(t,x) e^{f(x)} \Big) dx$$
$$\leq \Big(\int e^{-\frac{q\epsilon x^2}{2t}} dx \Big)^{\frac{1}{q}} \Big(\int \mathcal{Z}^{\mathbf{nw}}(t,x)^p e^{\frac{p\epsilon x^2}{2t}} e^{pf(x)} dx \Big)^{\frac{1}{p}}$$

Compute the Gaussian integral,

$$\mathbb{E}\big[\mathcal{Z}^f(t,0)^p\big] \leq \Big(\frac{2\pi t}{q\epsilon}\Big)^{\frac{p}{2q}} \mathbb{E}\Big[\int \mathcal{Z}^{\mathbf{nw}}(t,x)^p e^{\frac{p\epsilon x^2}{2t}} e^{pf(x)} dx\Big]$$

Recall $\mathbb{E}\big[\mathcal{Z}^{\mathbf{nw}}(t,x)^p\big] = \mathbb{E}\big[\mathcal{Z}^{\mathbf{nw}}(t,0)^p\big] e^{-\frac{px^2}{2t}}$,

$$\mathbb{E}\big[\mathcal{Z}^f(t,0)^p\big] \leq \Big(\frac{2\pi t}{q\epsilon}\Big)^{\frac{p}{2q}} \mathbb{E}\big[\mathcal{Z}^{\mathbf{nw}}(t,0)^p\big] \int e^{-\frac{p(1-\epsilon)x^2}{2t}} \mathbb{E}\big[e^{pf(x)}\big] dx$$

Upper bound, $p > 1$

$$\log \mathbb{E}\left[\mathcal{Z}^f(t,0)^p\right] \leq \frac{p}{2q} \log\left(\frac{2\pi t}{q\epsilon}\right) + \log \mathbb{E}\left[\mathcal{Z}^{\textbf{nw}}(t,0)^p\right] + \log \int e^{-\frac{p(1-\epsilon)x^2}{2t}} \mathbb{E}\left[e^{pf(x)}\right] dx$$

After dividing by t and letting $t \to \infty$, the first part goes to 0, second part goes to $\frac{p^3-p}{24}$ and third part is upper bounded by $g(p)$ using Condition 2.

Condition 2:

$$\liminf_{\epsilon \to 0} \lim_{t \to \infty} \frac{1}{t} \log\left(\int e^{-\frac{p(1-\epsilon)x^2}{2t}} \mathbb{E}\left[e^{p(1+\epsilon)f(x)}\right] dx\right) \leq g(p).$$

Upper bound, $p \in (0, 1]$

Due to sub-additivity, for $p \leq 1$

$$\Big(\sum_n a_n\Big)^p \leq \sum_n a_n^p$$

Then

$$\begin{aligned}\mathbb{E}\Big[\mathcal{Z}^f(t,0)^p\Big] &= \mathbb{E}\Big[\Big(\int \mathcal{Z}^{\mathbf{nw}}(t,x)e^{f(x)}dx\Big)^p\Big] \\ &\leq \sum_{n\in\mathbb{Z}} \mathbb{E}\Big[\big(\int_n^{n+1} \mathcal{Z}^{\mathbf{nw}}(t,x)e^{f(x)}dx\big)^p\Big].\end{aligned}$$

Need to compare $\mathcal{Z}^{\mathbf{nw}}(t,x)$ with $\mathcal{Z}^{\mathbf{nw}}(t,y)$ when x, y satisfy $|x-y| \leq 1$.

Increment tail bound

Proposition (Corwin, Ghosal, Hammond 19')

For any $\epsilon > 0$*, there exists* C, κ *s.t. for all* $t > 1$ *and* $s > 0$

$$\mathbb{P}\Big(\sup_{x \in [0, t^{\frac{1}{3}}]} \big(\mathcal{H}^{\mathbf{nw}}(t,x) - \mathcal{H}^{\mathbf{nw}}(t,0) - \frac{x^2}{2}\big) \geq s \Big) \leq C e^{-\kappa s^{\frac{9}{8}-\epsilon}}$$

$$\mathbb{P}\Big(\inf_{x \in [0, t^{\frac{1}{3}}]} \big(\mathcal{H}^{\mathbf{nw}}(t,x) - \mathcal{H}^{\mathbf{nw}}(t,0) + \frac{x^2}{2}\big) \leq -s \Big) \leq C e^{-\kappa s^{\frac{9}{8}-\epsilon}}$$

A useful lemma

Assume $\mathbb{P}\big(Y_t > s\big) \leq C\exp(-\kappa s^{1+\delta})$ for all t.

Lemma

Let X be a non-negative random variable. For any fixed $\epsilon > 0$, $\exists C$ s.t. for all t, $\mathbb{E}\big[Xe^{Y_t}\big] \leq C\Big(\mathbb{E}\big[X^{1+\epsilon}\big]\Big)^{\frac{1}{1+\epsilon}}$

Proof.

Y_t has uniform super-exponential tail, so for every $p > 0$, $\mathbb{E}\big[e^{pY_t}\big]$ is uniformly upper bounded. By Hölder inequality,
$$\mathbb{E}\big[Xe^{Y_t}\big] \leq \Big(\mathbb{E}\big[X^{1+\epsilon}\big]\Big)^{\frac{1}{1+\epsilon}}\Big(\mathbb{E}\big[e^{\frac{1+\epsilon}{\epsilon}Y_t}\big]\Big)^{\frac{\epsilon}{1+\epsilon}} \leq C\mathbb{E}\big[X^{1+\epsilon}\big]\Big)^{\frac{1}{1+\epsilon}}$$

Note that is Y_t has uniform super-exponential lower tail, then
$$\mathbb{E}\big[Xe^{Y_t}\big] \geq C\Big(\mathbb{E}\big[X^{1-\epsilon}\big]\Big)^{\frac{1}{1-\epsilon}}$$

Upper bound, $p \in (0,1]$

- $\mathbb{E}\big[\mathcal{Z}^f(t,0)^p\big] \le \sum_{n\in\mathbb{Z}} \underbrace{\mathbb{E}\Big[\Big(\int_n^{n+1} \mathcal{Z}^{\mathbf{nw}}(t,x) e^{f(x)} dx\Big)^p\Big]}_{\mathcal{I}_n}.$
- $\mathcal{J}_n(t) :=$
  $\sup_{x\in[n,n+1]} \big(\mathcal{H}^{\mathbf{nw}}(t,x) - \mathcal{H}^{\mathbf{nw}}(t,n) + \frac{(x-n)n}{t} - \frac{(x-n)^2}{2}\big)$
- $\mathcal{J}_n(t)$ is stationary in n and has super-exponential tail.

$\mathcal{Z}^{\mathbf{nw}}(t,x) \le C\mathcal{Z}^{\mathbf{nw}}(t,n)e^{\mathcal{J}_n(t)}$. Hence,

$$\mathcal{I}_n \le C\mathbb{E}\big[\mathcal{Z}^{\mathbf{nw}}(t,n)^p e^{p\mathcal{J}_n(t)}\big]\mathbb{E}\Big[\Big(\int_n^{n+1} e^{f(x)}dx\Big)^p\Big]$$

$$= Ce^{-\frac{pn^2}{2t}}\mathbb{E}\big[\mathcal{Z}^{\mathbf{nw}}(t,0)^p e^{p\mathcal{J}_0(t)}\big])\mathbb{E}\Big[\Big(\int_n^{n+1} e^{f(x)}dx\Big)^p\Big]$$

Upper bound, $p \in (0, 1]$

- $\mathbb{E}\Big[\mathcal{Z}^f(t,0)^p\Big] \leq \sum_{n\in\mathbb{Z}} \underbrace{\mathbb{E}\Big[\Big(\int_n^{n+1} \mathcal{Z}^{\mathsf{nw}}(t,x)e^{f(x)}dx\Big)^p\Big]}_{\mathcal{I}_n}.$
- $\mathcal{I}_n \leq Ce^{-\frac{pn^2}{2t}}\mathbb{E}\big[\mathcal{Z}^{\mathsf{nw}}(t,0)^p e^{p\mathcal{J}_0(t)}\big]\big)\mathbb{E}\Big[\big(\int_n^{n+1} e^{f(x)}dx\big)^p\Big]$
- This implies that

$$\mathbb{E}\big[\mathcal{Z}^f(t,0)^p\big] \leq C \underbrace{\mathbb{E}\big[\mathcal{Z}^{\mathsf{nw}}(t,0)^p e^{p\mathcal{J}_0(t)}\big]}_{\mathcal{A}_1(t)} \times$$

$$\underbrace{\Big(\sum_n e^{-\frac{pn^2}{2t}}\mathbb{E}\Big[\Big(\int_n^{n+1} e^{f(x)}dx\Big)^p\Big]\Big)}_{\mathcal{A}_2(t)}$$

Upper bound, $p \in (0, 1]$

$$\mathbb{E}\big[\mathcal{Z}^f(t,0)^p\big] \leq C \underbrace{\mathbb{E}\big[\mathcal{Z}^{\mathsf{nw}}(t,0)^p e^{p\mathcal{J}_0(t)}\big]}_{\mathcal{A}_1(t)} \times$$

$$\underbrace{\Big(\sum_n e^{-\frac{pn^2}{2t}} \mathbb{E}\Big[\Big(\int_n^{n+1} e^{f(x)} dx\Big)^p\Big]\Big)}_{\mathcal{A}_2(t)}$$

$$\mathcal{A}_1(t) \leq C\Big(\mathbb{E}\big[\mathcal{Z}^{\mathsf{nw}}(t,0)^{p(1+\epsilon)}\big]\Big)^{\frac{1}{1+\epsilon}}$$

$$\mathcal{A}_2(t) \leq C \int e^{-\frac{p(1-\epsilon)x^2}{2t}} \mathbb{E}\big[e^{p(1+\epsilon)f(x)}\big] dx$$

Recall Condition 2

$$\liminf_{\epsilon\to 0} \lim_{t\to\infty} \frac{1}{t} \log\Big(\int e^{-\frac{p(1-\epsilon)x^2}{2t}} \mathbb{E}\big[e^{p(1+\epsilon)f(x)}\big] dx\Big) \leq g(p).$$

Lower bound, $p > 0$

- $\mathbb{E}\big[\mathcal{Z}^f(t,0)^p\big] = \mathbb{E}\Big[\big(\int \mathcal{Z}^{\mathbf{nw}}(t,x)e^{f(x)}dx\big)^p\Big]$
- Condition 1 $\lim_{t\to\infty} \frac{1}{t}\sup(-\frac{px^2}{2t} + \log \mathbb{E}\big[e^{pf(x)}\big]) = g(p)$
- Assume the maximum is reached at $x_p(t)$, then

$$\lim_{t\to\infty} \frac{1}{t}\Big(-\frac{px_p(t)^2}{2t} + \log \mathbb{E}\big[e^{pf(x_p(t))}\big]\Big) = g(p).$$

- By localization around $x_p(t)$,

$$\mathbb{E}\big[\mathcal{Z}^f(t,0)^p\big] \geq \mathbb{E}\Big[\Big(\int_{x_p(t)}^{x_p(t)+1} \mathcal{Z}^{\mathbf{nw}}(t,x)e^{f(x)}dx\Big)^p\Big]$$

Lower bound, $p > 0$

- $\mathbb{E}\big[\mathcal{Z}^f(t,0)^p\big] \ge \mathbb{E}\Big[\big(\int_{x_p(t)}^{x_p(t)+1} \mathcal{Z}^{\mathbf{nw}}(t,x) e^{f(x)} dx\big)^p\Big]$
- $\mathcal{K}_p(t) = \inf_{x\in[x_p(t),x_p(t)+1]} \big(\mathcal{H}^{\mathbf{nw}}(t,x) - \mathcal{H}^{\mathbf{nw}}(t,x_p(t)) + \mathcal{O}(1)\big)$
- $\mathcal{K}_p(t) \overset{d}{=} \inf_{x\in[0,1]} \Big(\mathcal{H}^{\mathbf{nw}}(t,x) - \mathcal{H}^{\mathbf{nw}}(t,0) + \frac{x^2}{2}\Big)$ has uniform super-exponential lower tail
- $\mathcal{Z}^{\mathbf{nw}}(t,x) \ge C\mathcal{Z}^{\mathbf{nw}}(t,x_p(t))e^{\mathcal{K}_p(t)}, x \in [x_p(t), x_p(t)+1]$

$$\mathbb{E}\left[\mathcal{Z}^f(t,0)^p\right] \geq C\,\underbrace{\mathbb{E}\left[\mathcal{Z}^{\mathbf{nw}}(t,x_p(t))^p e^{p\mathcal{K}_p(t)}\right]}_{\mathcal{B}_1(t)}\underbrace{\mathbb{E}\left[\left(\int_{x_p(t)}^{x_p(t)+1} e^{f(x)}dx\right)^p\right]}_{\mathcal{B}_2(t)}$$

- We know that

$$\begin{aligned}\mathcal{B}_1(t) &\geq C\left(\mathbb{E}\left[\mathcal{Z}^{\mathbf{nw}}(t,x_p(t))^{p-\epsilon}\right]\right)^{\frac{p}{p-\epsilon}} \\ &= Ce^{-\frac{px_p(t)^2}{2t}}\left(\mathbb{E}\left[\mathcal{Z}^{\mathbf{nw}}(t,0)^{p-\epsilon}\right]\right)^{\frac{p}{p-\epsilon}}\end{aligned}$$

- $\mathcal{B}_2(t) \geq C\left(\mathbb{E}\left[e^{(p-\epsilon)f(x_p(t))}\right]\right)^{\frac{p}{p-\epsilon}}$

Lower bound, $p > 0$

$$\log \mathbb{E}\left[\mathcal{Z}^f(t,0)^p\right] \geq \log C + \frac{p}{p-\epsilon}\Big(\log \mathbb{E}\left[\mathcal{Z}^{\mathsf{nw}}(t,0)^{p-\epsilon}\right] - \frac{(p-\epsilon)x_p(t)^2}{2t} + \mathbb{E}\left[e^{(p-\epsilon)f(x_p(t))}\right]\Big).$$

Condition 1: $\lim_{t\to\infty} \frac{1}{t}\Big(- \frac{p x_p(t)^2}{2t} + \log \mathbb{E}\left[e^{pf(x_p(t))}\right]\Big) = g(p)$

The result follows after dividing t, let $t \to \infty$ then $\epsilon \to 0$.

Summary

1. We compute the Lyapunov exponent of the SHE with general initial condition and obtain the upper tail LDP of the KPZ equation.

2. The initial data includes deterministic sub-linear data and BM with drift.

3. We use (1) convolution formula; (2) narrow wedge SHE stationarity; (3) narrow wedge SHE Lyapunov exponent; (4) increment tail bound of the narrow wedge KPZ equation.

Future question

1. Can the conditions of initial data be relaxed?

2. LDP for the lower tail fo KPZ equation with general initial data. (Narrow wedge proved by [Tsai 18'] and [Cafasso-Claeys 19'])

3. Upper tail for half-space KPZ equation

$$\begin{cases} \partial_t \mathcal{H}^{\mathsf{hf}}(t,x) = \frac{1}{2}\partial_{xx}\mathcal{H}^{\mathsf{hf}}(t,x) + \xi(t,x)\mathcal{H}^{\mathsf{hf}}(t,x), & x > 0 \\ \partial_x \mathcal{H}^{\mathsf{hf}}(t,0) = A \end{cases}$$

$A = -\frac{1}{2}$ narrow wedge proved by [L. 20']. Other cases are unknown for now.

Thank you!