Recent progress on large deviations of the KPZ equation

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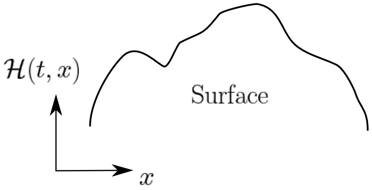
KPZ equation

- Introduced by Kardar-Parisi-Zhang (1986)

$$\partial_t \mathcal{H}(t,x) = \frac{1}{2}\partial_{xx}\mathcal{H}(t,x) + \frac{1}{2}\big(\partial_x \mathcal{H}(t,x)\big)^2 + \xi(t,x)$$

ξ is the space-time white noise



Stochastic heat equation (SHE)

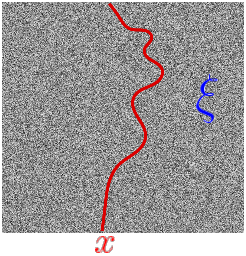
- A well-posed equation

$$\partial_t \mathcal{Z}(t,x) = \frac{1}{2}\partial_{xx}\mathcal{Z}(t,x) + \xi(t,x)\mathcal{Z}(t,x)$$

- Formally, by Feynman-Kac formula

$$\mathcal{Z}(t,x) = \mathbb{E}_x\Big[\exp\Big(\int_0^t \xi\big(s, B(t-s)\big)ds\Big)\mathcal{Z}(0, B(t))\Big]$$

Directed Polymer
in random environment



KPZ equation and SHE

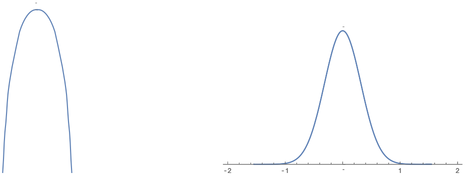
- KPZ equation

$$\partial_t \mathcal{H}(t,x) = \frac{1}{2}\partial_{xx}\mathcal{H}(t,x) + \frac{1}{2}\big(\partial_x \mathcal{H}(t,x)\big)^2 + \xi(t,x)$$

- Take $\mathcal{Z} = e^{\mathcal{H}}$, formally $\mathcal{Z}$ solves the SHE

$$\partial_t \mathcal{Z}(t,x) = \frac{1}{2}\partial_{xx}\mathcal{Z}(t,x) + \mathcal{Z}(t,x)\xi(t,x)$$

- Narrow wedge initial data: $\mathcal{Z}(0,x) = \delta_0(x)$, $\mathcal{Z}(t,x) \approx p(t,x)$



One point fluctuation

- [Amir-Corwin-Quastel 11]

$$\frac{\mathcal{H}(2t,0)+\frac{t}{12}}{t^{\frac{1}{3}}} \Longrightarrow \text{Tracy Widom GUE} \quad t\to\infty$$
$$\frac{\mathcal{H}(2t,0)+\log(\sqrt{4\pi t})}{t^{\frac{1}{4}}} \Longrightarrow \mathcal{N}(0,\frac{\pi}{2}) \qquad t\to 0$$

- What about large deviation principle (LDP)?
  - Long time regime Caffaso, Claeys, Corwin, Das, Ghosal, Tsai . . .
  - Short time regime
    - Physics: Kamenev, Katzav, Kolokolov, Korshunov, Le Doussal, Majumdar, Meerson, Sasorov, Schehr, Rosso, Vilenkin . . .
    - Math: ?

Long time regime

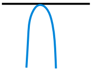
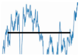
Long time regime

- It is known that

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(\mathcal{H}(2t,0) + \frac{t}{12} > st\Big) = -\Phi_+(s)$$
$$\lim_{t\to\infty} \frac{1}{t^2} \log \mathbb{P}\Big(\mathcal{H}(2t,0) + \frac{t}{12} < -st\Big) = -\Phi_-(s)$$

- [Das-Tsai 19] shows $\Phi_+(s) = \frac{4}{3}s^{\frac{3}{2}}$. [Tsai 18], [Caffaso-Claeys 19] identify $\Phi_-(s)$

Three initial conditions

	Fluctuation Limit	Upper tail LDP $\Phi_+(s)$
Narrow Wedge 	Tracy-Widom GUE	$\frac{4}{3}s^{\frac{3}{2}}$
Flat	Tracy-Widom GOE	?
BM 	Baik-Rains	?

Main Result

Let $\mathcal{H}^f$ denote the KPZ equation starting from $\mathcal{H}^f(0,\cdot) = f$

Theorem (Ghosal, L. 20)

If f is bounded, we have

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}(\mathcal{H}^f(2t,0) + \frac{t}{12} > st) = -\frac{4}{3}s^{\frac{3}{2}}, \quad s > 0$$

Theorem (Ghosal, L. 20)

If f is a two sided Brownian motion, we have

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}(\mathcal{H}^f(2t,0) + \frac{t}{12} > st) = -\frac{2}{3}s^{\frac{3}{2}}, \quad s > 0$$

In fact, Our theorem applies a more general class of initial data (e.g. $f(x) = |x|^\alpha, \alpha < 1$).

$$e^{\mathcal{H}} = \mathcal{Z}$$

KPZ equation

$$\partial_t \mathcal{H} = \frac{1}{2}\partial_{xx}\mathcal{H} + \frac{1}{2}(\partial_x \mathcal{H})^2 + \xi$$

SHE

$$\partial_t \mathcal{Z} = \frac{1}{2}\partial_{xx}\mathcal{Z} + \frac{1}{2}\xi \mathcal{Z}$$

Upper tail LDP

$$\mathbb{P}\big(\mathcal{H}(2t,0) + \frac{t}{12} > st\big) \approx e^{-t\Phi_+(s)}$$

Lyapunov exponent

$$\gamma_p = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[\mathcal{Z}(t,0)^p\right]$$

$$\Phi_+(s) \overset{\text{Legendre transform}}{\longleftrightarrow} 2\gamma_p + \frac{p}{12}$$

Lyapunov exponent

Definition

For $p > 0$, call γ_p to be p-th Lyapunov exponent of the SHE if

$$\gamma_p := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\big[\mathcal{Z}(t,0)^p\big]$$

- Lyapunov exponent and intermittency of various random fields were studied by Balan, Chen, Conus, Dalang, Hu, Huang, Janjigian, Khoshnevisan, Nualart, Tindel ...
- Integer Lyapunov exponent of SHE was studied by [Bertini-Cancrini 95], [Chen 15], [Corwin-Ghosal 18]
- Under narrow wedge initial data, [Das-Tsai 19] compute Lyapunov exponent $\gamma_p^{\textbf{nw}} = \frac{p^3-p}{24}$ for all $p > 0$

Result

Consider the SHE $\mathcal{Z}^f$ starting from the initial data $\exp(f)$

Theorem (Ghosal, L. 20)

(i). If f is bounded, $\gamma_p = \frac{p^3-p}{24}$ for all $p > 0$

(ii). If f is a two-sided Brownian motion, $\gamma_p = \frac{p^3}{6} - \frac{p}{24}$ for all $p > 0$

[Das-Tsai 19] computes $\gamma_p^{\textbf{nw}}$ using the exact formula

$$\mathbb{E}\Big[\exp\big(-s(\mathcal{Z}^{\textbf{nw}}(2t,0)+\frac{t}{12})\big)\Big] = \det(I - K_{s,t})$$

How to compute γ_p when we have general initial data?

Heuristic

- [Corwin-Hammond 16]

$$\mathcal{Z}^f(t,0) \overset{d}{=} \int \mathcal{Z}^{\mathbf{nw}}(t,x)e^{f(x)}dx$$

where f and $\mathcal{Z}^{\mathbf{nw}}$ are independent

- This implies $\mathbb{E}\big[\mathcal{Z}^f(t,0)^p\big] = \mathbb{E}\Big[\big(\int \mathcal{Z}^{\mathbf{nw}}(t,x)e^{f(x)}dx\big)^p\Big]$
- We have the approximation

$$\mathbb{E}\Big[\Big(\int \mathcal{Z}^{\mathbf{nw}}(t,x)e^{f(x)}dx\Big)^p\Big] \approx \mathbb{E}\Big[\int \mathcal{Z}^{\mathbf{nw}}(t,x)^p e^{pf(x)}dx\Big]$$

$$= \int \mathbb{E}\big[\mathcal{Z}^{\mathbf{nw}}(t,x)^p\big]\mathbb{E}\big[e^{pf(x)}\big]dx$$

$$= \mathbb{E}\big[\mathcal{Z}^{\mathbf{nw}}(t,0)^p\big] \int \mathbb{E}\big[e^{p(f(x)-\frac{x^2}{2t})}\big]dx \text{ (stationary of } \mathcal{Z}^{\mathbf{nw}}(t,x)e^{\frac{x^2}{2t}})$$

Heuristic

- $\log \mathbb{E}\big[\mathcal{Z}^f(t,0)^p\big] \approx \log \mathbb{E}\big[\mathcal{Z}^{\textbf{nw}}(t,0)^p\big] + \log \int \mathbb{E}\big[e^{p(f(x)-\frac{x^2}{2t})}\big]dx$

$$\gamma_p \quad = \quad \gamma_p^{\textbf{nw}} \quad + \quad g(p)$$

- $\gamma_p^{\textbf{nw}} = \frac{p^3-p}{24}$
- $g(p) = \lim_{t\to\infty} \frac{1}{t} \log \Big(\int \mathbb{E}\big[e^{p(f(x)-\frac{x^2}{2t})}\big]dx \Big)$
  - $g(p) = 0$ if f is bounded
  - $g(p) = \frac{p^3}{8}$ if f is two-sided Brownian motion
- The heuristic also applies to other function-valued initial condition

Short time regime

Short time LDP of the KPZ equation

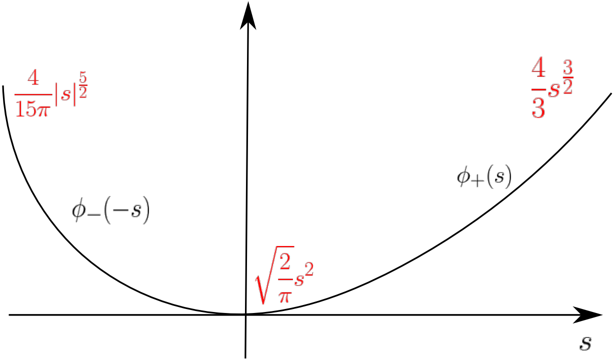
- Recall $\frac{\mathcal{H}(2t,0)+\log(\sqrt{4\pi t})}{t^{\frac{1}{4}}} \Longrightarrow \mathcal{N}(0, \frac{\pi}{2})$ as $t \to 0$.
- For $s > 0$,

$$\lim_{t\to 0} \sqrt{t}\log \mathbb{P}\big(\mathcal{H}(2t,0) + \log(\sqrt{4\pi t}) > s\big) = -\phi_+(s)$$
$$\lim_{t\to 0} \sqrt{t}\log \mathbb{P}\big(\mathcal{H}(2t,0) + \log(\sqrt{4\pi t}) < -s\big) = -\phi_-(s)$$

- Studied in the physics literature [Kolokolov-Korshunov 07, 09] [Le Doussal-Majumdar-Schehr-Rosso 16] [Kamenev-Meerson-Sasarov 16], [Katzav-Meerson-Vilenkin 16]...
- Mathematically, the existence of $\phi_+(s)$ and $\phi_-(s)$ is not known

Rate function

Predicted in the physics literature, the rate function has the asymptotics as follows.



Main Result

Theorem (L., Tsai 20+)

There exist ϕ_+, ϕ_- *s.t. for any* $s \geq 0$,

$$\lim_{t \to 0} \sqrt{t} \log \mathbb{P}\big(\mathcal{H}(2t, 0) + \log(\sqrt{4\pi t}) > s\big) = -\phi_+(s)$$

$$\lim_{t \to 0} \sqrt{t} \log \mathbb{P}\big(\mathcal{H}(2t, 0) + \log(\sqrt{4\pi t}) < -s\big) = -\phi_-(s)$$

In addition,

$$\lim_{s \to 0} \frac{\phi_\pm(s)}{s^2} = \frac{1}{\sqrt{2\pi}}, \qquad \lim_{s \to \infty} \frac{\phi_-(s)}{s^{\frac{5}{2}}} = \frac{4}{15\pi}$$

Conjecture

We have $\lim_{s \to \infty} \phi_+(s)/s^{\frac{3}{2}} = \frac{4}{3}$

Possible methods of proof

- Exact formula approach: [Le Doussal-Majumdar-Schehr-Rosso 16]
- Weak noise theory approach: [Kamenev-Meerson-Sasarov 16], [Katzav-Meerson-Vilenkin 16]
  - Consider scaling $\mathcal{H}_\epsilon(t,x) = \mathcal{H}(\epsilon^2 t, \epsilon x) + \log \epsilon$. Then

$$\partial_t \mathcal{H}_\epsilon = \frac{1}{2}\partial_{xx}\mathcal{H}_\epsilon + \frac{1}{2}(\partial_x \mathcal{H}_\epsilon)^2 + \sqrt{\epsilon}\xi$$

  - Study the LDP of the trajectory of $\mathcal{H}_\epsilon$ on $[0,2] \times \mathbb{R}$
  - To understand the short time LDP problem, only need the LDP of $\mathcal{H}_\epsilon(2,0) + \log(\sqrt{4\pi})$ as $\epsilon \to 0$

Weak noise theory

- For $\rho \in L^2([0,2] \times \mathbb{R})$,

$$-\epsilon \log \mathbb{P}\big(\sqrt{\epsilon}\xi \approx \rho\big) \approx \frac{1}{2}\|\rho\|^2, \qquad \epsilon \to 0$$

Note that

$$\partial_t \mathcal{H}_\epsilon - \frac{1}{2}\partial_{xx}\mathcal{H}_\epsilon - \frac{1}{2}(\partial_x \mathcal{H}_\epsilon)^2 = \sqrt{\epsilon}\xi.$$

This implies that

$$-\epsilon \log \mathbb{P}\big(\mathcal{H}_\epsilon \approx h\big) \approx \frac{1}{2}\|\underbrace{\partial_t h - \frac{1}{2}\partial_{xx} h - \frac{1}{2}(\partial_x h)^2}_{\downarrow\ \rho}\|^2$$

Heuristic

- Let $\rho = \partial_t h - \frac{1}{2}\partial_{xx} h - \frac{1}{2}(\partial_x h)^2$, then

$$-\epsilon \log \mathbb{P}\big(\mathcal{H}_\epsilon \approx h\big) \approx \frac{1}{2}\|\rho\|^2, \quad \epsilon \to 0.$$

  - We want to find $-\epsilon \log \mathbb{P}\big(\mathcal{H}_\epsilon(2,0) \approx -\lambda\big) \approx ?$ as $\epsilon \to 0$
  - Given $h(2,0) = -\lambda$, minimize

$$\frac{1}{2}\int_0^2 \int \big(\partial_t h - \frac{1}{2}\partial_{xx} h - \frac{1}{2}(\partial_x h)^2\big)^2 dt dx.$$

- By calculus of variation, we get

$$\partial_t \rho + \frac{1}{2}\partial_{xx}\rho = \partial_x\big(\rho \partial_x h\big)$$
$$\rho = \partial_t h - \frac{1}{2}\partial_{xx} h - \frac{1}{2}(\partial_x h)^2$$

Heuristic

- We want the LDP of $\mathcal{H}_\epsilon(2,0) = -\lambda$ as $\lambda \to 0$ or ∞.
  - $\lambda \to 0$ is easy.
  - $\lambda \to \infty$, we scale $h(t,x) \to \lambda h(t, \lambda^{-\frac{1}{2}}x), \rho \to \lambda\rho(t, \lambda^{-\frac{1}{2}}x)$
- Under this scaling, we have

$$\partial_t h - \frac{1}{2\lambda}\partial_{xx} h - \frac{1}{2}(\partial_x h)^2 = \rho$$
$$\partial_t \rho + \frac{1}{2\lambda}\partial_{xx}\rho = \partial_x\big(\rho\partial_x h\big)$$

Heuristic

- Remove the viscosity as $\lambda \to \infty$, we have

$$\partial_t h - \frac{1}{2}(\partial_x h)^2 = \rho$$
$$\partial_t \rho = \partial_x \big(\rho \partial_x h\big)$$

- Solve the PDEs, one gets

$$\rho_*(t,x) = -\frac{1}{2\pi} r(t)(1 - \frac{x^2}{\ell(t)^2})\mathbf{1}_{\{|x| \le \ell(t)\}}, \quad \frac{1}{2}\|\rho_*\|^2 = \frac{4}{15\pi}$$

- Recall we expect $\phi_-(\lambda) \approx \frac{4}{15\pi}\lambda^{\frac{5}{2}}$, $\lambda \to \infty$.

Our approach

- How to argue that one can remove the viscosity?
- Why

$$\rho_*(t,x) = -\frac{1}{2\pi} r(t)(1 - \frac{x^2}{\ell(t)^2})\mathbf{1}_{\{|x|\le \ell(t)\}}$$

is indeed the minimizer as $\lambda \to \infty$?

- Our approach:
  - We look at SHE.
  - We obtain the LDP of SHE in terms of the Feynman Kac's formula.
  - We use Varadhan's lemma to obtain the asymptotic.

Our Approach

- Consider $\mathcal{Z}_\epsilon = e^{\mathcal{H}_\epsilon}$, then

$$\partial_t \mathcal{Z}_\epsilon = \frac{1}{2}\partial_{xx}\mathcal{Z}_\epsilon + \sqrt{\epsilon}\xi\mathcal{Z}_\epsilon, \quad \mathcal{Z}_\epsilon(0,\cdot) = \delta_0(\cdot)$$

- Recall Feynman-Kac formula

$$\mathcal{Z}_\epsilon(t,x) = \mathbb{E}_x\Big[\exp\Big(\int_0^t \xi\big(s, B(t-s)\big)ds\Big)\delta_0(B(t))\Big]$$

$$= \underbrace{\mathbb{E}_{x\to 0}\Big[\exp\Big(\int_0^t \sqrt{\epsilon}\xi(s, B_{\mathsf{b}}(s))ds\Big)\Big]}_{\mathcal{J}(\sqrt{\epsilon}\xi)} p(t,x)$$

Theorem (L. Tsai 20+)

$\mathcal{Z}_\epsilon(\cdot,\cdot)/p(\cdot,\cdot)$ *satisfies a LDP with rate* ϵ^{-1} *and a good rate function*

$$I(f) = \inf\Big\{\frac{1}{2}\|\rho\|^2 : \mathcal{J}(\rho) = f, \rho \in L^2([0,2]\times\mathbb{R})\Big\}$$

One point rate function

We want the LDP of $\mathcal{H}_\epsilon(2,0) + \log\sqrt{4\pi}$

Corollary (L. Tsai 20+)

Let $\mathcal{S}(\rho) = \big(\log \mathcal{J}(\rho)\big)(2,0) + \log\sqrt{4\pi}$. *We have for* $\lambda \geq 0$,

$$\phi_-(\lambda) = \inf\left\{\frac{1}{2}\|\rho\|^2 : \mathcal{S}(\rho) \leq -\lambda\right\}$$

$$\phi_+(\lambda) = \inf\left\{\frac{1}{2}\|\rho\|^2 : \mathcal{S}(\rho) \geq \lambda\right\}$$

Asymptotics of the rate function

- Want to prove $\lim_{\lambda\to\infty} \frac{\phi_-(\lambda)}{\lambda^{\frac{5}{2}}} = \frac{4}{15\pi}$
- Scale $\rho \to \lambda\rho(t, \lambda^{-\frac{1}{2}}x)$, then

$$\phi_-(\lambda) = \lambda^{\frac{5}{2}} \inf\left\{\frac{1}{2}\|\rho\|^2 : \mathcal{S}_\lambda(\rho) \le -1\right\}$$

where $\mathcal{S}_\lambda(\rho) = \lambda^{-1}\log\mathbb{E}\Big[\exp\big(\lambda\int_0^2 \rho(s, \lambda^{-\frac{1}{2}}B_{\mathsf{b}}(s))ds\big)\Big]$

- $\rho_*(t,x) = -\frac{1}{2\pi}r(t)(1-\frac{x^2}{\ell(t)^2})\mathbf{1}_{\{|x|\le\ell(t)\}}, \quad \frac{1}{2}\|\rho_*\|^2 = \frac{4}{15\pi}$
- (1). $\lim_{\lambda\to\infty}\mathcal{S}_\lambda(\rho_*) = -1$ (2). ρ_* is the minimizer

LDP in the LDP

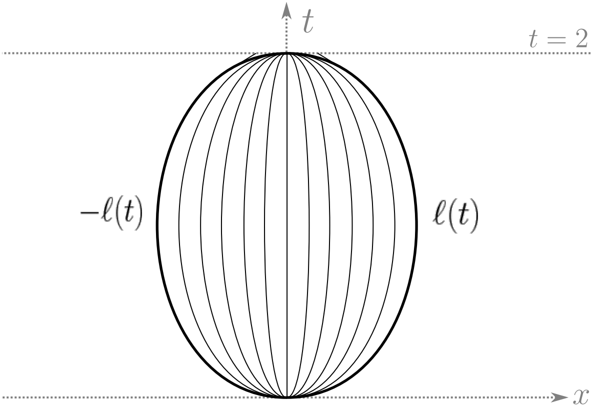
- By Varadhan's lemma

$$\lim_{\lambda\to\infty} \lambda^{-1} \log \mathbb{E}_{0\to 0}\Big[\exp\big(\lambda \int_0^2 \rho_*(s, \lambda^{-\frac{1}{2}} B_{\mathsf{b}}(s))ds\big)\Big]$$
$$= \sup_{x(0)=x(2)=0} \Big(\int_0^2 \rho_*(s, x(s)) - \frac{1}{2}\dot{x}(s)^2 ds\Big)$$

- We can find the geodesics from the Euler-Lagrange Equation

$$\ddot{x}(s) + \partial_x \rho_*(s, x(s)) = 0$$

Geodesics



$$x(t) = \alpha\ell(t), \alpha \in [-1, 1]$$

Summary

- Long time: We find the upper tail rate function for a class of initial datas which includes the flat and Brownian initial data
- Short time: For narrow wedge initial data, we prove a LDP and show a crossover in the lower tail rate function from exponent 2 to $\frac{5}{2}$

Thank you!