The stochastic telegraph equation limit of stochastic higher spin six vertex model

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Abstract

In this project, we prove that the stochastic telegraph equation arises as a scaling limit of the stochastic higher spin six vertex (SHS6V) model with general spin $I/2, J/2$. This extends results of Borodin and Gorin which focused on the $I = J = 1$ six vertex case and demonstrates the universality of the stochastic telegraph equation in this context. We also provide an functional extension of the central limit theorem obtained in [BG19, ST19]. The main idea is to generalize the four point relation established in [BG19], using fusion.

(Stochastic) telegraph equation

The *telegraph equation* is a hyperbolic PDE given by

$$\begin{cases} u_{XY}(X,Y) + \beta_1 u_Y(X,Y) + \beta_2 u_X(X,Y) = f(X,Y) \\ u(X,0) = \chi(X), \quad u(0,Y) = \psi(Y). \end{cases} \tag{1}$$

The equation admits a unique explicit solution. When the boundary equals zero, it has a simple form

$$u(X,Y) = \int_0^X \int_0^Y \mathcal{R}(X,Y,x,y) f(x,y)\,dx\,dy,$$

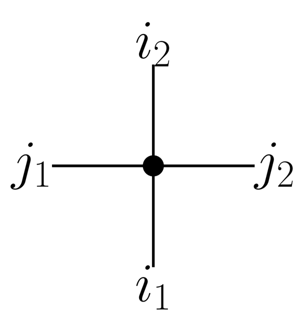
where $\mathcal{R}$ is the *Riemann function*.

If $f(X,Y) = \sqrt{\theta(X,Y)}\eta(X,Y)$, where η is the space-time white noise and θ is some L^2 function, we call (1) the *stochastic telegraph equation*.

Stochastic higher spin six vertex model

The SHS6V model is a four parameter family of models that encompass many known KPZ integrable systems. For a vertex, assign the weight $L(i_1, j_1; i_2, j_2)$ to the configuration indexed by $(i_1, j_1, i_2, j_2) \in \mathbb{Z}^4_{\geq 0}$. Take i_1, j_1 as the number of input lines and i_2, j_2 as the output. L is a stochastic matrix with row and column indexed by $(i_1, j_1), (i_2, j_2) \in \{0, 1, \dots, I\} \times \{0, 1, \dots, J\}$.

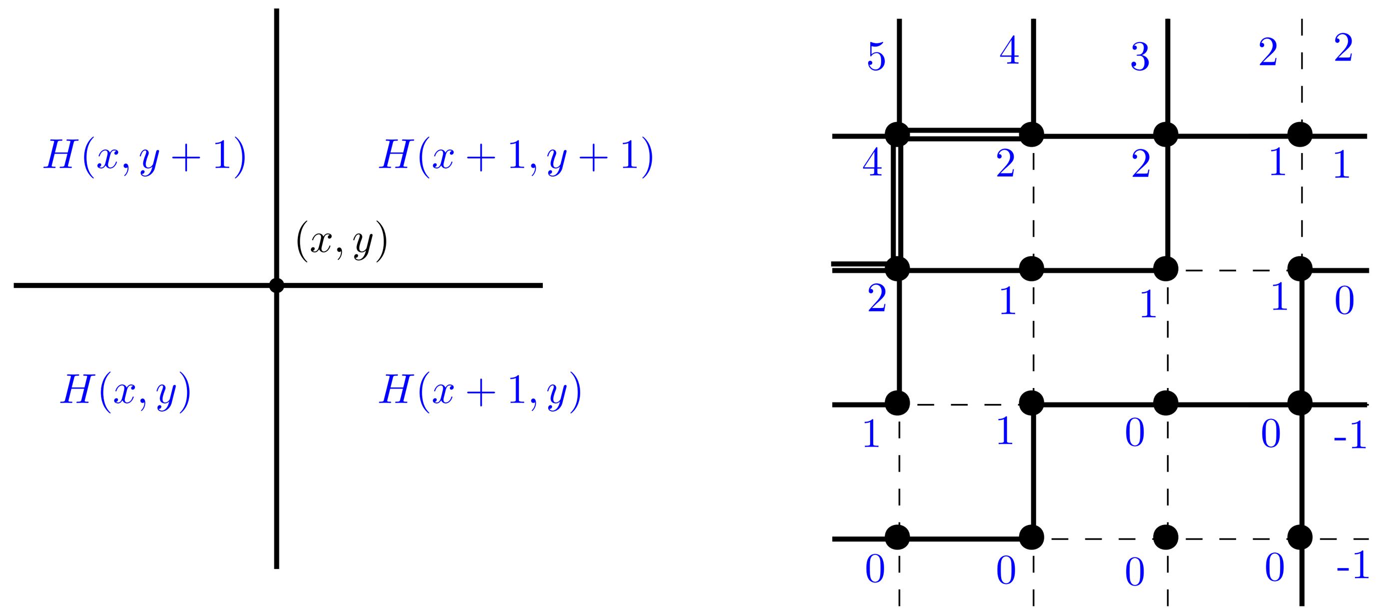
We provide the weight for $J = 1$ (always set $\nu = q^{-I}$). For general J, the weight also admits an explicit (though complicated) expression. It can also be defined implicitly using fusion.



Vertex	$0\,\overset{m}{\underset{m}{+}}\,0$	$0\,\overset{m-1}{\underset{m}{+}}\,1$	$1\,\overset{m}{\underset{m}{+}}\,1$	$1\,\overset{m+1}{\underset{m}{+}}\,0$
Weight	$\frac{1+\alpha q^m}{1+\alpha}$	$\frac{\alpha(1-q^m)}{1+\alpha}$	$\frac{\alpha+\nu q^m}{1+\alpha}$	$\frac{1-\nu q^m}{1+\alpha}$

Height function

We study the SHS6V model on the first quadrant, viewing it as a stochastic path ensemble going to the upright. Let $H : \mathbb{Z}^2_{\geq 0} \to \mathbb{R}$ be a height function defined in the way that: Set $H(0,0) = 0$, when we move across i vertical lines from left to right, H decreases by i. Moving across j horizontal lines from bottom to top makes H increase by j. Extend $H(x,y)$ to all $(x,y) \in \mathbb{R}^2_{\geq 0}$ by linear interpolation.



Scaling and Four point relation

Consider the scaling: Fix I, J, β_1, β_2, take $q = e^{\frac{\beta_1 - \beta_2}{L}}$ and define α through $e^{-\frac{J\beta_2}{L}} = \frac{1+\alpha q^J}{1+\alpha}$, then let $L \to \infty$.

The four point was first obtained in [BG19] for the stochastic six vertex model. We generalize it the SHS6V model, define

$$\begin{aligned} \xi(x+1,y+1) = q^{H(x+1,y+1)} &- b_1 q^{H(x,y+1)} \\ &- b_2 q^{H(x+1,y)} + (b_1 + b_2 - 1) q^{H(x,y)}, \end{aligned}$$

where $b_1 = \frac{\alpha+\nu}{1+\alpha}$, $b_2 = \frac{1+\alpha q^J}{1+\alpha}$. Let $\mathcal{F}(x,y)$ be sigma algebra generated by $H(i,j)$ for $i \leq x$ or $j \leq y$ and define

$$\Delta_x = q^{H(x+1,y)} - q^{H(x,y)}, \quad \Delta_y = q^{H(x,y+1)} - q^{H(x,y)}.$$

Theorem

(i). $\mathbb{E}\big[\xi(x+1,y+1) \,|\, \mathcal{F}(x,y)\big] = 0$,

(ii). $\mathbb{E}\big[\xi(x+1,y+1)^2 \,|\, \mathcal{F}(x,y)\big] = L^{-1}(\beta_1+\beta_2)\Delta_x\Delta_y +$
$JL^{-2}(\beta_2-\beta_1)\beta_2 q^{H(x,y)}\Delta_x + IL^{-2}(\beta_1-\beta_2)\beta_1 q^{H(x,y)}\Delta_y + \mathbf{R}(x,y)$,

There exists a constant C s.t. the error term $|\mathbf{R}(x,y)| \leq CL^{-4}$.

Borodin and Gorin proved (ii) for the stochastic six vertex model without the error term $\mathbf{R}(x,y)$, which is no longer true for the SHS6V model. Also it is only under our scaling that $\mathbf{R}$ is negligible.

Main Result

Theorem

- (Hydrodynamic limit)

$$\frac{1}{L}H(Lx, Ly) \xrightarrow{p} \mathbf{h}(x,y) \quad \text{in } C(\mathbb{R}^2_{\geq 0}),$$

where $\mathbf{h}$ is the solution to (let $\mathsf{q} = e^{\beta_1 - \beta_2}$)

$$\frac{\partial^2}{\partial x \partial y}\mathsf{q}^{\mathbf{h}(x,y)} + J\beta_2 \frac{\partial}{\partial x}\mathsf{q}^{\mathbf{h}(x,y)} + I\beta_1 \frac{\partial}{\partial y}\mathsf{q}^{\mathbf{h}(x,y)} = 0.$$

- (Functional central limit theorem)

$$\sqrt{L}\big(q^{H(Lx,Ly)} - \mathbb{E}[q^{H(Lx,Ly)}]\big) \Rightarrow \phi(x,y) \quad \text{in } C(\mathbb{R}^2_{\geq 0}),$$

$\phi(x,y)$ solves an explicit stochastic telegraph equation.

Methods of Proof

Four point relation:

The main idea is to use fusion. First, prove the equality for $J = 1$ directly. For general J, decompose the vertex into a column of $J = 1$ vertices. Applying the $J = 1$ four point relation for these vertices and summing up these identities, a telescoping property arises and we conclude (i). For (ii), we need additionally the property of our scaling which says, with high probability, the line keeps flowing in the same direction.

Main result:

The four point relation (i) suggests $q^{H((x,y)}$ should satisfy a discrete version of the telegraph equation, which implies the hydrodynamic limit. The functional CLT follows from

(1). The finite dimensional weak convergence (2). Tightness

The finite dimensional weak convergence is due to (ii) and the martingale CLT as in [BG19]. The tightness follows from the moment bound for the increment, which can be expressed as a summation of martingale increment. Using Burkholder inequality and an estimate of the joint moments for $\xi(x,y)$ yields the desired moment bound.

References

[BG19] A. Borodin and V. Gorin: A stochastic telegraph equation from the six-vertex model, *Ann. Probab.* (2019).

[ST19] H. Shen and L. Tsai: Stochastic Telegraph Equation Limit for the Stochastic Six Vertex Model, *Proc. Amer. Math. Soc.* (2019).

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