KPZ equation limit of stochastic higher spin six vertex model

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KPZ equation

KPZ equation [Kardar-Parisi-Zhang]

$$\partial_t \mathcal{H}(t,x) = \frac{\delta}{2}\partial_{xx}\mathcal{H}(t,x) + \frac{\kappa}{2}(\partial_x \mathcal{H}(t,x))^2 + \sqrt{D}\xi(t,x).$$

$\xi(t,x)$ is the space-time white noise.

Hopf-Cole transform $\mathcal{Z}(t,x) = e^{\frac{\kappa}{\delta}\mathcal{H}(t,x)}$, $\mathcal{Z}(t,x)$ is the solution of the stochastic heat equation (SHE)

$$\partial_t \mathcal{Z}(t,x) = \frac{\delta}{2}\partial_{xx}\mathcal{Z}(t,x) + \frac{\kappa}{\delta}\sqrt{D}\mathcal{Z}(t,x)\xi(t,x).$$

KPZ equation

Consider $\mathcal{H}_\epsilon(t,x) = \epsilon^{-z}\mathcal{H}(\epsilon^{-b}t, \epsilon^{-1}x)$,

$$\partial_t \mathcal{H}_\epsilon = \frac{\delta}{2}\epsilon^{2-b}\partial_x^2 \mathcal{H}_\epsilon + \frac{\kappa}{2}\epsilon^{-z+2-b}(\partial_x \mathcal{H}_\epsilon)^2 + \epsilon^{z+\frac{1}{2}-\frac{b}{2}}\sqrt{D}\xi.$$

There is no b, z such that the KPZ equation is invariant.

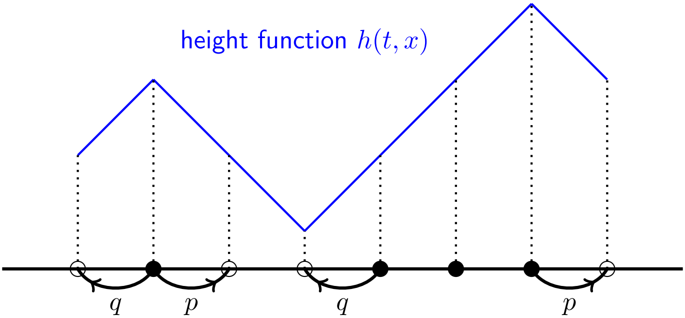
Invariant if we allow to tune the parameter

- $b = 2, z = \frac{1}{2}, \kappa \to \sqrt{\epsilon}\kappa$,
- $b = 3, z = 1, \delta \to \epsilon\delta, \kappa \to \epsilon^2\kappa$.

Discrete models $\Rightarrow$ KPZ equation if we tune the parameters properly while doing scaling.

ASEP converges to KPZ equation

[Bertini-Giacomin]



Take $p - q = \sqrt{\epsilon}$, as $\epsilon \downarrow 0$

$$\sqrt{\epsilon}\big(h(\epsilon^{-2}t, \epsilon^{-1}x) - a_\epsilon t\big) \Rightarrow \mathcal{H}(\cdot,\cdot).$$

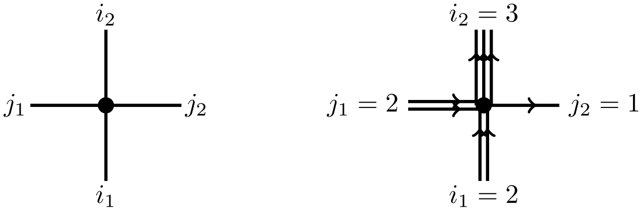
Other results: ASEP(q, j) [Corwin-Shen-Tsai], Open ASEP [Corwin-Shen], [Parekh], higher spin exclusion process [Corwin-Tsai], stochastic six vertex model [Corwin-Ghosal-Shen-Tsai] . . .

The stochastic higher spin six vertex model

Four-parameter family of quantum integrable systems introduced by [Corwin-Petrov]. It is on top of hierarchy of KPZ class integrable system.

The stochastic higher spin six vertex model

Vertex configuration: $i_1, i_2 \in \{0, 1, \ldots, I\}$ and $j_1, j_2 \in \{0, 1, \ldots, J\}$, $i_1 + j_1 = i_2 + j_2$.

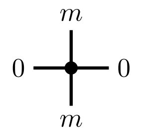
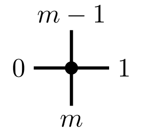
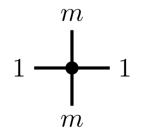
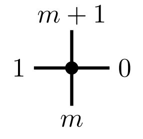


weight $L(i_1, j_1; i_2, j_2)$

i_1, j_1 vertical and horizontal input lines

i_2, j_2 vertical and horizontal output lines

$J = 1$ vertex weight

Vertex				
Weight	$\frac{1+\alpha q^m}{1+\alpha}$	$\frac{\alpha(1-q^m)}{1+\alpha}$	$\frac{\alpha+\nu q^m}{1+\alpha}$	$\frac{1-\nu q^m}{1+\alpha}$

where $\nu = q^{-I}$.

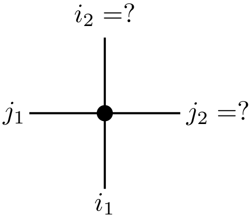
General J vertex weight is defined through fusion [Kirillov-Reshetikhin]. Closed expression given by [Mangazeev], [Corwin-Petrov]

$$L(i_1, j_1; i_2, j_2) = \mathbf{1}_{i_1+j_1=i_2+j_2} q^{\frac{2j_1-j_1^2}{4} - \frac{2j_2-j_2^2}{4} + \frac{i_2^2+i_1^2}{4} + \frac{i_2(j_2-1)+i_1 j_1}{2}}$$
$$\times \frac{\nu^{j_1-i_2} \alpha^{j_2-j_1+i_2} (-\alpha\nu^{-1}; q)_{j_2-i_1}}{(q;q)_{i_2} (-\alpha; q)_{i_2+j_2} (q^{J+1-j_1}; q)_{j_1-j_2}} {}_4\bar{\phi}_3(\dots).$$

Four parameters α, q, I, J.

Vertex weight

Under certain choice of α, q, $L(i_1, j_1; i_2, j_2)$ is a stochastic matrix with row and column indexed by
$(i_1, j_1), (i_2, j_2) \in \{0, 1, \ldots, I\} \times \{0, 1, \ldots, J\}$.



Given input (i_1, j_1), choose output (i_2, j_2) according to a probability measure $L(i_1, j_1; \cdot, \cdot)$.

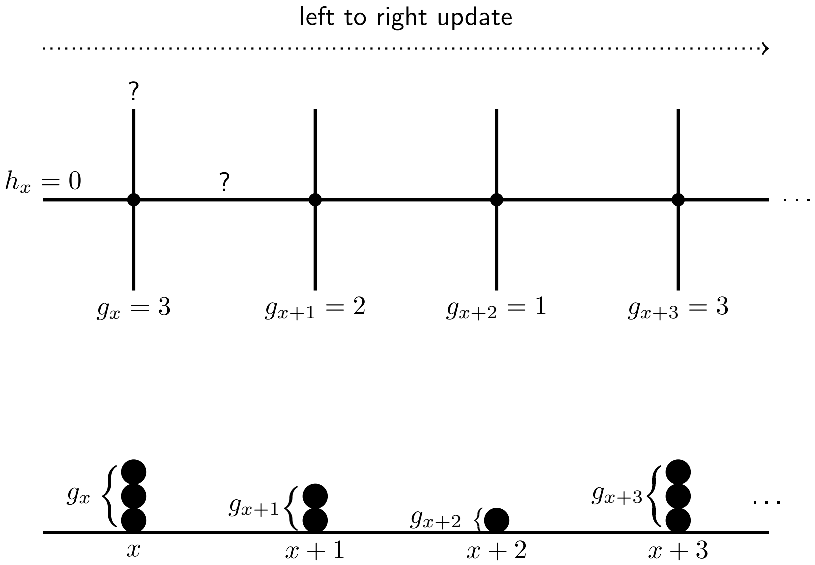
Particle system Interpretation

Discrete time Markov process $\vec{g}(t)$ on the space of left-finite particle configuration

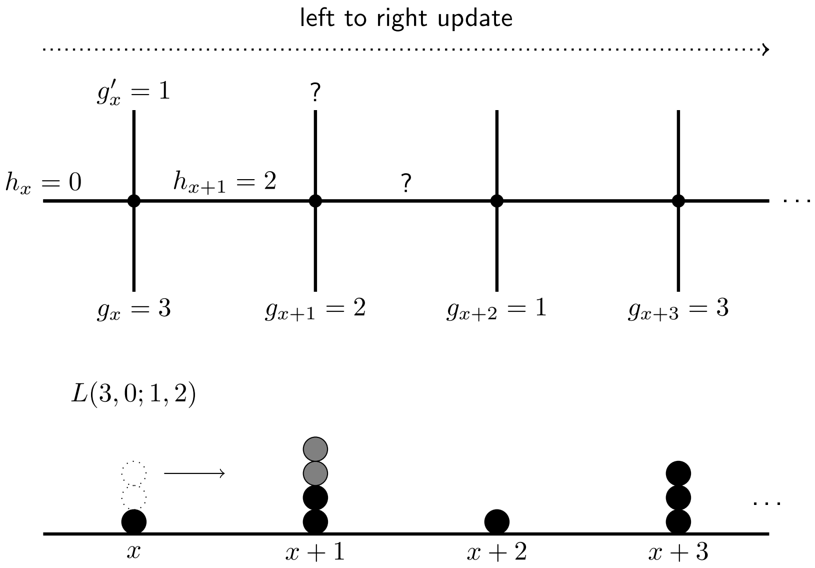
$$\{\vec{g} = (\ldots, g_{-1}, g_0, g_1 \ldots) : \text{ all } g_i \in \{0, 1, \ldots, I\}$$
$$\text{and there exists } x \in \mathbb{Z} \text{ such that } g_i = 0 \text{ for all } i < x\}$$

Specify the update procedure from $\vec{g}$ to $\vec{g}'$

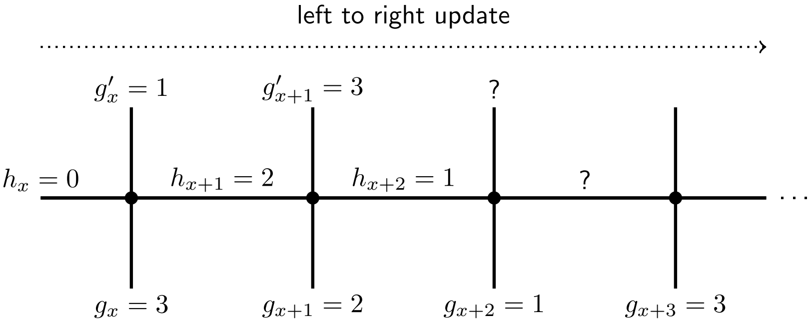
Particle system interpretation



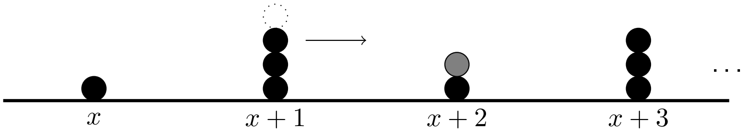
Particle system interpretation



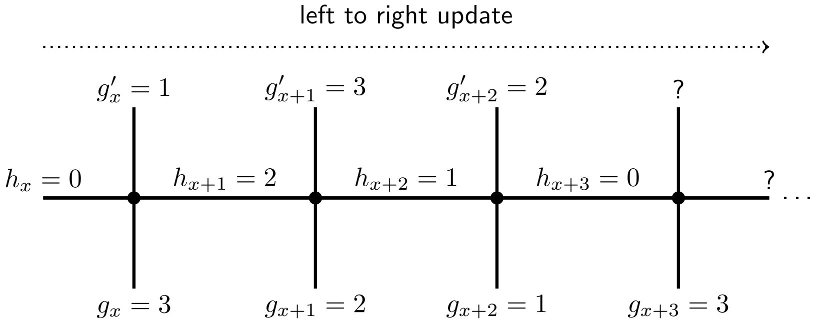
Particle system interpretation



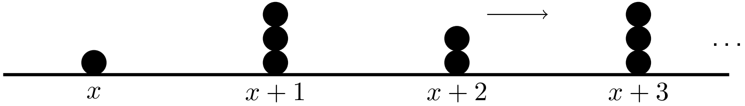
$$L(3,0;1,2) \times L(2,2;3,1)$$



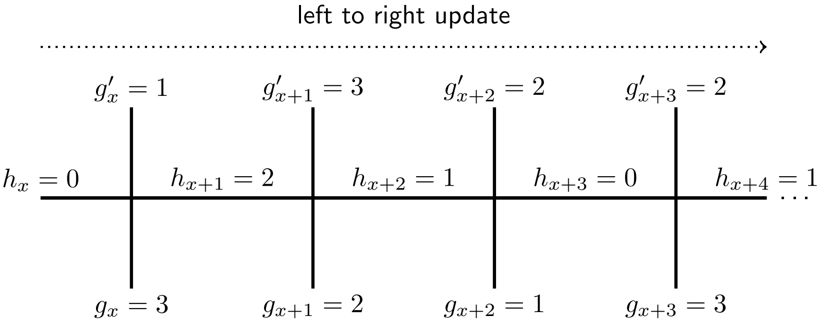
Particle system interpretation



$$L(3,0;1,2) \;\times\; L(2,2;3,1) \;\times\; L(1,1;2,0)$$



Particle system interpretation



$$L(3,0;1,2) \;\times\; L(2,2;3,1) \;\times\; L(1,1;2,0) \;\times\; L(3,0;2,1) \;\times \cdots$$

Particle system interpretation

- Call the discrete time-homogeneous Markov process $\vec{g}(t)$ the stochastic higher spin six vertex model.

- Fusion: One can define a time-inhomogeneous Markov process $\vec{\eta}(t)$ via replacing $L_{\alpha,q}^{I,J}$ with $L_{\alpha(t),q}^{I,1}$, where $\alpha(t) = \alpha q^{mod_J(t)}$. Then

$$\big(\vec{g}(t), t \geq 0\big) = \big(\vec{\eta}(Jt), t \geq 0\big) \qquad \text{in law}$$

- Define height function

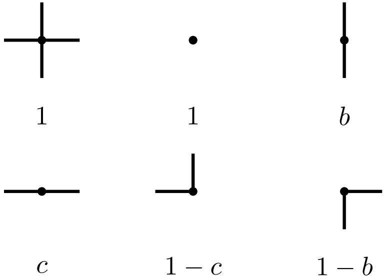
$$N(t,x) = N_x(\vec{g}(t)) - N_0(\vec{g}(0)), \qquad N(0,0) = 0,$$

where $N_x(\vec{g}) = \sum_{y \leq x} g_y$.

Weakly asymmetric scaling

Four parameters α, q, I, J.

- Fix $I, J \in \mathbb{Z}_{\geq 1}, b \in (\frac{I+J-2}{I+J-1}, 1)$, set $q = e^{\sqrt{\epsilon}}$ and define α through $b = \frac{1+\alpha q}{1+\alpha}$.
- When $I = J = 1$ (S6V model), fix b and scale $c = e^{-\sqrt{\epsilon}} b$



Main result

Theorem (L. 2019)

Under weakly asymmetric scaling, fix particle density $\rho \in (0, I)$, *as* $\epsilon \downarrow 0$

$$\sqrt{\epsilon}\Big(N(\epsilon^{-2}t, \epsilon^{-1}x + \epsilon^{-2}\mu_\epsilon t) - \rho(\epsilon^{-1}x + \epsilon^{-2}\mu_\epsilon t)\Big) - t\log\lambda_\epsilon \Rightarrow \mathcal{H}(\cdot,\cdot),$$

where

$$\mathcal{H}(t,x) = \frac{\delta}{2}\partial_x^2\mathcal{H}(t,x) + \frac{\kappa}{2}\big(\partial_x\mathcal{H}(t,x)\big)^2 + \sqrt{D}\xi(t,x),$$

δ, κ, D *are explicit.*

Former results

- [Corwin-Tsai] Another regime of model which allows infinite number of particles.
- [Corwin-Ghosal-Shen-Tsai] Stochastic six vertex model $(I = J = 1)$.

Sketch of the proof

Microscopic Hopf-Cole transform

- Let $Z(t,x) = \lambda^t q^{-N(t,x+\mu t)+\rho(x+\mu t)}$. The discrete SHE

$$Z(t+1,x) = (p * Z(t))(x) + M(t,x)$$

  p is the transition probability of a mean zero random walk, $M(t,x)$ is a martingale increment ($\mathbb{E}\left[M(t,x)\middle|\mathcal{F}(t)\right] = 0$).
- The existence of discrete SHE guaranteed by Markov duality.

Microscopic Hopf-Cole transform

- Need to show

$$Z(t+1) = p * Z(t) + M \Rightarrow \partial_t \mathcal{Z} = \frac{\kappa}{2}\partial_{xx}\mathcal{Z} + \frac{\kappa\sqrt{D}}{\delta}\xi\mathcal{Z}.$$

- Quadratic variation $[M]$ converges to $\frac{\kappa^2 D}{\delta^2}\mathcal{Z}^2$. For $x_1 \le x_2$

$$\mathbb{E}\big[M(t,x_1)M(t,x_2)\big|\mathcal{F}(t)\big] = \left(q^\rho \frac{\nu+\alpha}{1+\alpha}\right)^{|x_1-x_2|} \Theta_1(t,x_1)\Theta_2(t,x_1)$$

$$\epsilon^{-\frac{1}{2}}\Theta_1(t,x) := c_0 Z(t,x) + \sum_{i=1}^{\infty} c_i \epsilon^{-\frac{1}{2}} \nabla Z(t,x-i),$$

$$\epsilon^{-\frac{1}{2}}\Theta_2(t,x) := c_0' Z(t,x) + \sum_{i=1}^{\infty} c_i' \epsilon^{-\frac{1}{2}} \nabla Z(t,x-i),$$

where $\nabla Z(t,x) := Z(t,x+1) - Z(t,x)$.

Self-averaging

After averaging over a long time interval of order $\mathcal{O}(\epsilon^{-2})$

1. If $x_1 < x_2$, $\epsilon^{-1}\nabla Z(t,x_1)\nabla Z(t,x_2)$ vanishes.
2. $(\epsilon^{-\frac{1}{2}}\nabla Z(t,x))^2 - \frac{\rho(I-\rho)}{I}Z(t,x)^2$ vanishes.

Note that

$$\epsilon^{-\frac{1}{2}}\nabla Z(t,x) \approx (\rho - \mathbf{g}_x(t))Z(t,x),$$

$\epsilon^{-\frac{1}{2}}\nabla Z(t,x)$ same order as $Z(t,x)$, but vanishes after averaging.

Self-averaging

GOAL: Show $\epsilon^{-1}\nabla Z(t,x_1)\nabla Z(t,x_2)$ vanishes after averaging if $x_1 < x_2$

- Control $\mathbb{E}\big[\epsilon^{-1}\nabla Z(t,x_1)\nabla Z(t,x_2)\big|\mathcal{F}(s)\big]$.
- Two particle Markov duality [Corwin-Petrov]

$$\mathbb{E}\big[Z(t,x_1)Z(t,x_2)\big|\mathcal{F}(s)\big] = \sum_{y_1\le y_2} \mathbf{V}\big((x_1,x_2),(y_1,y_2),t,s\big)Z(s,y_1)Z(s,y_2)$$

  $\mathbf{V}$ is a tilted two particle transition probability of the model.
- If $x_1 < x_2$,

$$\mathbb{E}\big[\epsilon^{-1}\nabla Z(t,x_1)\nabla Z(t,x_2)\big|\mathcal{F}(s)\big] = \epsilon^{-1}\sum_{y_1\le y_2} \nabla_{x_2}\nabla_{x_1}\mathbf{V}\big((x_1,x_2),(y_1,y_2),t,s\big)Z(s,y_1)Z(s,y_2).$$

Estimate of two particle transition probability

Integral formula of $\mathbf{V}$: Bethe ansatz function [Corwin-Petrov] + generalized Fourier theory [Borodin-Corwin-Petrov-Sasamoto].

$$\begin{aligned}\mathbf{V}\big((x_1,x_2);(y_1,y_2),t,s\big) &= \oint\oint \prod_{i=1}^{2} \mathfrak{D}(z_i)^{t-s} z_i^{x_i-y_i} \frac{dz_i}{2\pi\mathbf{i}} \\ &- \oint\oint \mathfrak{F}(z_1,z_2) \prod_{i=1}^{2} \mathfrak{D}(z_i)^{t-s} z_i^{x_{3-i}-y_i} \frac{dz_i}{2\pi\mathbf{i}} \\ &+ \text{Residue of } \oint\oint \mathfrak{F}(z_1,z_2) \prod_{i=1}^{2} \mathfrak{D}(z_i)^{t-s} z_i^{x_{3-i}-y_i} \frac{dz_i}{2\pi\mathbf{i}}.\end{aligned}$$

- Steepest descent analysis.
  - $\mathbf{V}\big((x_1,x_2),(y_1,y_2),t,s\big)$: temporal decay order $\frac{1}{\sqrt{t-s}}$
  - $\nabla\mathbf{V}\big((x_1,x_2),(y_1,y_2),t,s\big)$ temporal decay order $\frac{1}{t-s}$
  - $\nabla_{x_1}\nabla_{x_2}\mathbf{V}\big((x_1,x_2),(y_1,y_2),t,s\big)$ temporal decay order $\frac{1}{(t-s)^{\frac{3}{2}}}$

Self-averaging

GOAL: Show $(\epsilon^{-1}\nabla Z(t,x))^2 - \frac{\rho(I-\rho)}{I} Z(t,x)^2$ vanishes after averaging.

Note that

$$\left(\epsilon^{-\frac{1}{2}}\nabla Z(t,x)\right)^2 \approx (g_x(t)-\rho)^2 Z(t,x)^2.$$

The stationary distribution of the model with density ρ is a product measure $\bigotimes \pi_\rho$ [Imamura-Mucciconi-Sasamoto], where

$$\pi_\rho(k) = \frac{(\chi,q)_\infty}{(\chi\nu,q)_\infty}\frac{(\nu,q)_k}{(q,q)_k}\chi^k, \quad k \in \{0,1,\ldots,I\}.$$

Then

$$\mathbb{E}_{\pi_\rho}\left[(g_x(t)-\rho)^2\right] = \mathsf{Var}\left[\pi_\rho\right] \approx \frac{\rho(I-\rho)}{I}.$$

Self-averaging, $I = 1$

GOAL: Show $(\epsilon^{-\frac{1}{2}}\nabla Z(t,x))^2 - \frac{\rho(I-\rho)}{I} Z(t,x)^2$ vanishes after averaging.

$$\left(\epsilon^{-\frac{1}{2}}\nabla Z(t,x)\right)^2 \approx (g_x(t)-\rho)^2 Z(t,x)^2.$$

When $I = 1$ [Corwin-Ghosal-Shen-Tsai], using $g_x(t)^2 = g_x(t)$,

$$\begin{aligned}(g_x(t)-\rho)^2 Z(t,x)^2 &= \left(\rho^2 - (2\rho-1)g_x(t)\right)Z(t,x)^2 \\ &= \rho(1-\rho)Z(t,x)^2 + (2\rho-1)\epsilon^{-\frac{1}{2}}\nabla Z(t,x)Z(t,x),\end{aligned}$$

which implies

$$\left(\epsilon^{-\frac{1}{2}}\nabla Z(t,x)\right)^2 - \rho(1-\rho)Z(t,x)^2 \approx (2\rho-1)\epsilon^{-\frac{1}{2}}\nabla Z(t,x)Z(t,x).$$

Self-averaging, $I \geq 2$

Need to estimate $\mathbb{E}\Big[(\epsilon^{-\frac{1}{2}}\nabla Z(t,x))^2 - \frac{\rho(I-\rho)}{I} Z(t,x)^2 \Big| \mathcal{F}(s)\Big]$.

Two particle version of Markov duality [Kuan]

$$\mathbb{E}\big[D(t,x_1,x_2)\big|\mathcal{F}(s)\big] = \sum_{y_1 \leq y_2} D(s,y_1,y_2)\mathbf{V}\big((x_1,x_2),(y_1,y_2),t,s\big),$$

where approximately

$$D(t,x_1,x_2) = \begin{cases} Z(t,x_1)Z(t,x_2)\big(I - g_{x_1}(t)\big)\big(I - 1 - g_{x_1}(t)\big) & \text{if } x_1 = x_2, \\ \frac{I-1}{I} Z(t,x_1)Z(t,x_2)\big(I - g_{x_1}(t)\big)\big(I - g_{x_2}(t)\big) & \text{if } x_1 < x_2. \end{cases}$$

The expression of D is different depending on whether $x_1 = x_2$.

Self-averaging, $I \geq 2$

- Rewrite into duality functional

$$\begin{aligned}
&\epsilon^{-1}(\nabla Z(t,x))^2 - \frac{\rho(I-\rho)}{I} Z(t,x)^2 \\
&\approx (g_x(t)-\rho)^2 Z(t,x)^2 - \frac{\rho(I-\rho)}{I} Z(t,x)^2, \\
&\approx D(t,x,x) - \frac{(I-1)(I-\rho)}{I} Z(t,x)^2 + \epsilon^{-\frac{1}{2}} \nabla Z(t,x) Z(t,x).
\end{aligned}$$

- Apply duality

$$\begin{aligned}
&\mathbb{E}\left[D(t,x,x) - \frac{(I-1)(I-\rho)}{I} Z(t,x)^2 \middle| \mathcal{F}(s) \right], \\
&= \sum_{y_1 \leq y_2} \mathbf{V}(\,\cdot\,) \left(D(s,y_1,y_2) - \frac{(I-1)(I-\rho)}{I} Z(s,y_1) Z(s,y_2) \right), \\
&\approx \sum_{y_1 < y_2} \mathbf{V}(\,\cdot\,) \left(c_1 \epsilon^{-\frac{1}{2}} \nabla Z(s,y_1) + c_2 \epsilon^{-\frac{1}{2}} \nabla Z(s,y_2) \right).
\end{aligned}$$

- Summation by part

$$\sum_{y<x} f(y) \nabla g(y) = f(x-1) g(x) - \sum_{y<x} \nabla f(y-1) g(y).$$

Thank you!