Examples of Markov Chain in Math Modelling

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Outline

- Galton-Watson process
- Continuous time Markov chain
- Yule process, Birth and death process, contact process

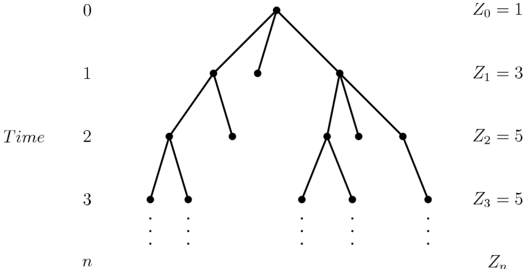
Galton-Watson process

In the mid 19th century several aristocratic families in Victorian England realized that their family names could become extinct. Was it just unfounded paranoia, or did something real prompt them to come to this conclusion? They decided to ask around, and Sir Francis Galton (a polymath, anthropologist, eugenicist, tropical explorer, geographer, inventor, meteorologist, proto-geneticist, psychometrician and statistician" and half-cousin of Charles Darwin) posed the following question (1873, Educational Times): How many male children (on average) must each generation of a family have in order for the family name to continue in perpetuity?

An answer came from Reverend Henry William Watson soon after, and the two wrote a joint paper entitled the probability of extinction of families (1874).

The model is now called Galton-Watson process (or branching process).

- Each individual independently generates a random number of offspring according to a certain probability distribution on $\mathbb{Z}_{\geq 0}$. (E.g. $0, 1, 2, 3$ with probability $\frac{1}{4}$)



- Let Z_n be the number of individual at time n. Suppose $Z_0 = 1$. We are interested in the probability q that the processes **die out** at some time. We call q the extinction probability.

Galton-Watson process

- When q will be 0?
- When q will be 1?
- If $q \in (0, 1)$, how can we find q?

Galton-Watson process

- Let p_n be the probability that an individual generates n offspring. Consider the average number of offspring

$$\mu = \sum_{n=0}^{\infty} n p_n$$

- If $\mu < 1$, then $q = 1$. In other words, the process will always die out.
- If $\mu > 1$, then $q < 1$. How to find out the value of q?
- Idea of recursion. Starting with $Z_0 = 1$, the extinction probability is q. Starting with n individuals, the extinction probability is q^n (why?). So we have the equation

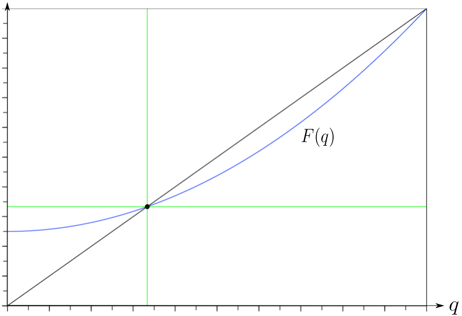
$$q = \sum_{k=0}^{\infty} p_k q^k$$

Idea of recursion

- Define the generating function

$$F(q) = \sum_{k=0}^{\infty} p_k q^k$$

- Find the root $F(q) = q$ for $q \in (0, 1)$



Summary

- When q will be 0?
  - if and only if $p_0 = 0$
- When q will be 1?
  - if $\mu < 1$
- If $q \in (0, 1)$, how can we find q?
  - q is the unique root of $F(q) = q$.

Continuous time Markov Chain

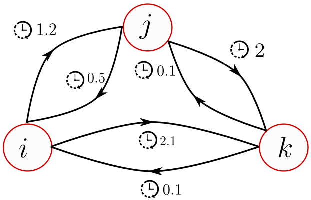
- To introduce other interesting models, we want our time to be continuous.

- Discrete time Markov chain: Time $t = 0, 1, 2, \dots$
  - specify transition probability p_{ij} from state i to j

- Continuous time Markov chain: Time $t \in [0, \infty)$
  - specify transition rate q_{ij} from state i to j (In a small time Δt, the probability to jump from i to j is given by $q_{ij}\Delta t$).

- Example of transition matrix:

$$\mathbf{Q} = \begin{array}{c} \\ i \\ j \\ k \end{array} \begin{array}{c} \begin{array}{ccc} i & j & k \end{array} \\ \begin{pmatrix} -3.3 & 1.2 & 2.1 \\ 0.5 & -2.5 & 2 \\ 0.1 & 0.1 & -0.2 \end{pmatrix} \end{array}$$



Kolmogorov's equation

- Let $p_{ij}(t)$ denote the transition probability to go from i (time 0) to j (time t). As a continuous time version of Chapman Kolmogorov equation, we have

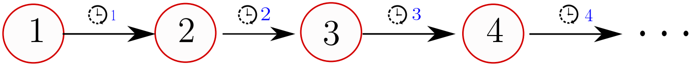
$$\frac{d}{dt}p_{ij}(t) = \sum_k p_{ik}(t)q_{kj}$$

- As we will see, this is a useful tool for computing $p_{ij}(t)$.

- Let us look at some examples.

Yule process

- Studied by British statistician Undy Yule. Also called pure birth process.
- A cell will split into 2 with rate 1 (independent of age and between different cells). How to characterize the number of cells at time t?
- Let N_t denote the number of cells at time t, $N_0 = 1$.
- The transition rate of N_t from n to $n+1$ is given by n.



- What is $\mathbb{P}(N_t = n)$?

Yule process

- Using Kolmogorov's equation, one get for every $n \geq 0$,

$$\frac{d}{dt} p_{n+1}(t) = n(p_n(t) - p_{n+1}(t))$$

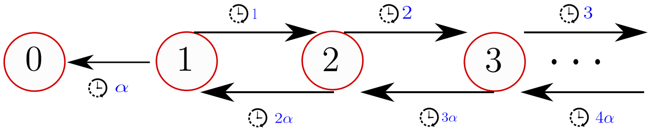
- $p_1(t) = e^{-t}$. Solving the Kolmogorov's equation, we get

$$p_n(t) = e^{-t}(1 - e^{-t})^{n-1}$$

- Question. If $N_0 = m$, what is $p_n(t)$?

$$p_n(t) = \binom{n-1}{m-1} e^{-mt}(1 - e^{-mt})^{n-m}$$

- In addition to splitting, each cell has rate α to die out. The dynamic of chain becomes



The transition rate of N_t from n to $n+1$ is given by n (birth), while the rate to go from n to $n-1$ is given by $n\alpha$ (death).

- Set $X_0 = m$, we want to compute extinction probability q_m. Clearly $q_0 = 1$, note that

$$q_m = \frac{1}{1+\alpha} q_{m+1} + \frac{\alpha}{1+\alpha} q_{m-1} \qquad \text{(why?)}$$

Then we have

$$q_{m+1} - q_m = \alpha(q_m - q_{m-1})$$

Birth and Death process

- $q_{m+1} - q_m = \alpha(q_m - q_{m-1})$

- If $\alpha > 1$. the only possible case is that $q_1 - q_0 = 0$. So $q_m = 1$ for all $m \geq 0$ (why?).

- If $\alpha = 1$. The only possible case is that $q_1 - q_0 = 0$. So $q_m = 1$ for all $m \geq 0$.

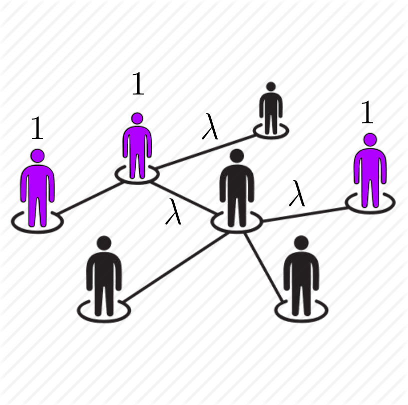
- If $\alpha < 1$. A reasonable assumption is that $\lim_{m\to\infty} q_m = 0$. Let $b = q_1 - q_0$, $q_n - q_{n-1} = b\alpha^{n-1}$,

$$-1 = \sum_{i=1}^{\infty}(q_i - q_{i-1}) = \sum_{i=1}^{\infty} b\alpha^{i-1} = \frac{b}{1-\alpha}$$

So $b = \alpha - 1$. $q_n = 1 + \sum_{i=1}^{n}(q_i - q_{i-1}) = \alpha^n$

Contact process

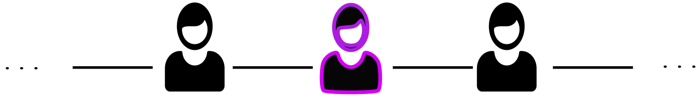
- The contact processes were first introduced and studied by Ted Harris. It is a nice model for the spread of an infection.



- Each individual is either healthy or infected. The infection spreads with λ. When everyone is healthy, no one will get infected again.

Contact process

- Let N_t be the number of infected individuals at time t.
- Let q be the probability of the event $\{N_t > 0 \text{ for all } t \geq 0\}$.
- When the graph is finite, eventually everyone will be healthy $(q = 0)$. What happens when the graph becomes infinite?
- Harris originally studies the contact process on $\mathbb{Z}^d$. Here we consider $d = 1$. If at the beginning, only the person at 0 is infected.



- Phase transition (Harris 1974). There exists a λ_c such that when $\lambda < \lambda_c$, $q = 0$. When $\lambda > \lambda_c$, $q > 0$.

Reference

- H. W. Watson and Francis Galton **On the Probability of the Extinction of Families** *The Journal of the Anthropological Institute of Great Britain and Ireland Vol. 4 (1875), pp. 138-144*
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- Rick Durrett **Probability: theory and examples** *Vol.49 Cambridge university press (2019)*
- Thomas M. Liggett **Interacting particle systems.** *Vol. 276. Springer Science Business Media (2012).*