

Markov duality for stochastic six vertex model

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Algebraic Duality Methods in Probability

Markov duality

Definition







Let $X(t), Y(t)$ be two time-homogeneous Markov processes on state spaces \mathbf{X}, \mathbf{Y} . We say $X(t)$ and $Y(t)$ are dual with respect to the function $D : \mathbf{X} \times \mathbf{Y} \rightarrow \mathbb{R}$, if

$$\mathbb{E}_x[D(X(t), y)] = \mathbb{E}_y[D(x, Y(t))] \quad (*)$$

for any $x \in \mathbf{X}, y \in \mathbf{Y}$ and t .

S6V model

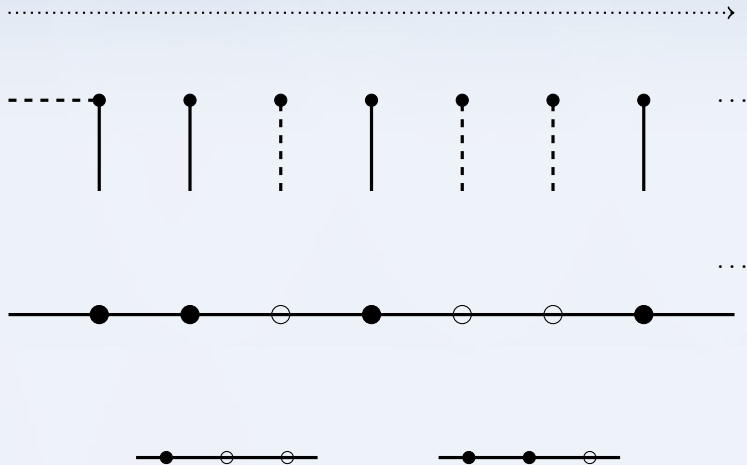
- Studied by [Gwa-Sphon 92]. Tiling model with six possible vertex configurations.

Type	I	II	III	IV	V	VI
Configuration						
Weight	1	1	b_2	$1 - b_2$	b_1	$1 - b_1$

- We view it as an interacting particle system on \mathbb{Z} . Two parameters $0 < b_1, b_2 < 1$.

Update rule

left to right update



Height function



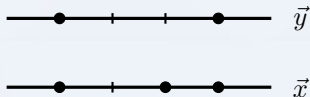
Height function $N_y(\vec{x}) = \#\{\text{particles which stay on the left or at } y\}$

Result

Theorem (L. 19)

The S6V model $X(t) = (x_1(t) < \cdots < x_k(t))$ and the space reversed S6V model $Y(t) = (y_1(t) > \cdots > y_m(t))$ are dual w.r.t.

$$H(\vec{x}, \vec{y}) = \prod_{i=1}^m q^{-N_{y_i}(\vec{x})} \mathbf{1}_{y_i}(\vec{x}).$$

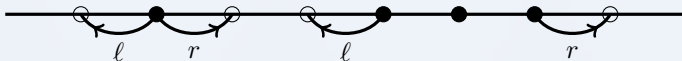


Questions

- **Why we expect that this is a duality function?**
- How do we prove this duality?
- What is the use of this duality?

ASEP

- The ASEP on \mathbb{Z} .



S6V model to ASEP

- Let $b_1 \rightarrow \varepsilon b_1$, $b_2 \rightarrow \varepsilon b_2$. Scale time $t \rightarrow \varepsilon^{-1}t$ and consider a moving frame with speed 1, [Borodin-Corwin-Gorin 16], [Aggarwal 17] showed that $X(\varepsilon^{-1}t) - \varepsilon^{-1}t$ converges to ASEP with left jump rate b_1 and right jump rate b_2 .



ASEP duality

- Let $X(t) = (x_1(t) < \cdots < x_k(t))$ denotes ASEP with left jump rate ℓ and right jump rate r . $Y(t) = (y_1(t) > \cdots > y_m(t))$ be the ASEP with left jump rate r and right jump rate ℓ . Define $q = \frac{\ell}{r}$.
- [Schütz 97] proved that $X(t)$ and $Y(t)$ are dual w.r.t.

$$H(\vec{x}, \vec{y}) = \prod_{i=1}^m q^{-N_{y_i}(\vec{x})} \mathbf{1}_{y_i}(\vec{x}).$$

- [Borodin-Corwin-Sasamoto 14] and [Corwin-Petrov 16] showed that both ASEP and the S6V model are self-dual w.r.t.

$$G(\vec{x}, \vec{y}) = \prod_{i=1}^m q^{-N_{y_i}(\vec{x})}.$$

- We wonder whether H is a duality function for the S6V model.

Theorem (L. 19)

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$$H(\vec{x}, \vec{y}) = \prod_{i=1}^m q^{-N_{y_i}(\vec{x})} \mathbf{1}_{y_i}(\vec{x}).$$

- Why we expect that this is a duality function? ✓
- **How do we prove this duality?**

Proof idea

- It suffices to show

$$\mathbb{E}_{\vec{x}}[H(X(1), \vec{y})] = \mathbb{E}_{\vec{y}}[H(\vec{x}, Y(1))]. \quad (*)$$

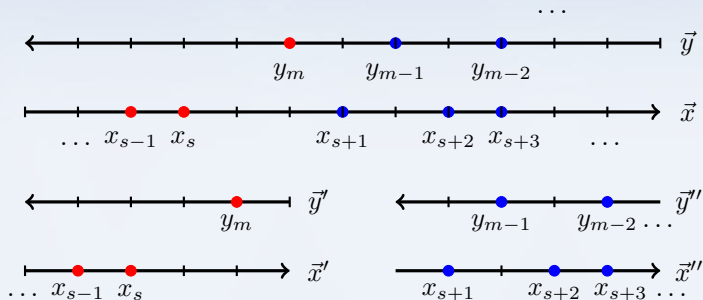
- We prove (*) by using an inductive argument over the particle number.
- Assume that \vec{x} has k particles and \vec{y} has m particles.

Induction basis: $k = 1$ or $m = 1$.

Induction argument: If (*) holds for all $k' + m' < k + m$, show (*) also holds for (k, m)

Inductive proof

- If $y_m \notin \vec{x}$



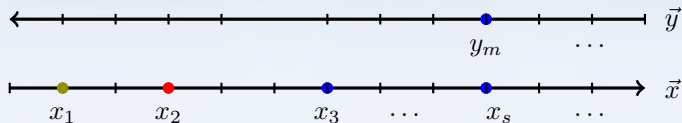
- We have

$$\mathbb{E}_{\vec{x}} [H(X(1), \vec{y})] = q^{-s(m-1)} \mathbb{E}_{\vec{x}'} [H(X(1), \vec{y}')] \mathbb{E}_{\vec{x}''} [H(X(1), \vec{y}'')],$$

$$\mathbb{E}_{\vec{y}} [H(\vec{x}, Y(1))] = q^{-s(m-1)} \mathbb{E}_{\vec{y}'} [H(\vec{x}', Y(1))] \mathbb{E}_{\vec{y}''} [H(\vec{x}'', Y(1))].$$

Inductive proof

- If $y_m \in \vec{x}$ (assume that $y_m > x_2$)



- $\vec{x}' = \vec{x} - \{x_1\}$ and $\vec{x}'' = \vec{x} - \{x_1, x_2\}$.

$$\begin{aligned}L_1 &= \mathbb{E}_{\vec{x}'}[H(X(1), \vec{y})], & L_2 &= \mathbb{E}_{\vec{x}''}[H(X(1), \vec{y})]. \\R_1 &= \mathbb{E}_{\vec{y}}[H(\vec{x}', Y(1))], & R_2 &= \mathbb{E}_{\vec{y}}[H(\vec{x}'', Y(1))].\end{aligned}$$

We have

$$\begin{aligned}\mathbb{E}_{\vec{x}}[H(X(1), \vec{y})] &= f(L_1, L_2), \\ \mathbb{E}_{\vec{y}}[H(\vec{x}, Y(1))] &= f(R_1, R_2).\end{aligned}$$

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$$H(\vec{x}, \vec{y}) = \prod_{i=1}^m q^{-N_{y_i}(\vec{x})} \mathbf{1}_{y_i}(\vec{x}).$$

- Why we expect that this is a duality function? ✓
- How do we prove this duality? ✓
- **What is the use of this duality?**

Moment formula

- Let $\mathcal{B}(t)$ be the semigroup of $Y(t)$.

- We have

$$\mathbb{E}_{\vec{x}}[H(X(t), \vec{y})] = \mathbb{E}_{\vec{y}}[H(\vec{x}, Y(t))] = \mathcal{B}(t)H(\vec{x}, \vec{y})$$

Fourier analysis [Borodin-Corwin-Petrov-Sasamoto 15] showed $H(\vec{x}, \cdot)$ can be decomposed into eigenfunctions of $\mathcal{B}(t)$, this yields

$$\mathbb{E}_{\vec{x}}[H(X(t), \vec{y})] = m\text{-fold contour integrals.}$$

- This finally implies an integral formula of $\mathbb{E}[q^{-mN_x(t)}]$ for the S6V model starting from the step-initial data [Borodin-Corwin-Gorin 16].

KPZ limit

- The idea was carried out in [Corwin-Ghosal-Shen-Tsai 20].
- One particle duality [Corwin-Petrov 16] shows that

$$\mathbb{E} \left[q^{-N(t+1,x)} \mid \mathcal{F}(t) \right] = \sum_y p(x-y) q^{-N(t,x)}.$$

Let $Z(t, x)$ denote a version of $q^{-N(t,x)}$, we have the discrete SHE

$$dZ(t, x) = (\tilde{p} \star Z(t))(x) + M(t, x).$$

Use duality [Corwin-Petrov 16] and [L. 19] to prove that the discrete SHE converges to its continuum

$$\partial_t \mathcal{Z} = \frac{1}{2} \nu \partial_{xx} \mathcal{Z} + \sqrt{D} \xi \mathcal{Z}.$$

Thank you!