Stat 23400 Lecture 1: Introduction to Probability

Course logistics

Objective & Goals

- Understand the basic principles of probability, distributions of random variables, expectation and variance, and the Central Limit Theorem.
- Apply estimation and testing methods to analyze single variables or the relationship between two variables in order to understand natural phenomena and make data-based decisions.
- Use statistical software to summarize data numerically and visually, and to perform data analysis.
- Interpret results correctly, effectively, and in context without relying on statistical jargon.
- Critique data-based claims and evaluate data-based decisions.

General Info

- Time: Tuesday and Thursday

Sec 1	11:00 am - 12:20 pm	Instructor:	Yier Lin
Sec 2	12:30 pm - 1:50 pm		Daniel Xiang
Sec 3	2:00 pm - 3:30 pm		Chih-Hsuan Wu

- Location: Eckhart 133 in person. Please **only** attend the section that you enroll in.
- **Textbooks:**
  - OpenIntro Statistics, 4th edition, by Diez, Barr, and Cetinkaya-Rundel.
  - Modern Mathematical Statistics with Applications. 2nd edition, by Jay L. Devore and Kenneth N. Berk.

  Both are reserved on canvas through the university library. Free downloads are available online.

Logistics

- The lecture materials will be uploaded on canvas after each lecture.
- R Lab Sessions: Links to the Lab sessions will be provided on canvas each week. Our Lead TA will also cover the Lab materials during his/her office hour on Zoom.
- Assignments: All posted on canvas, and submitted through Gradescope.
- Exams: May 3 (Tue), will take place in person in the classroom. Please email to the instructor as soon as possible if you need special accommodations

Need Help?

- Live Office Hours on Zoom: Links to the office hours are available on canvas.
- Ed Discussion: Questions are encouraged be posted on Ed Discussion and will (in most cases) be answered the same day if posted before 9pm during a weekday, and next day if posted after 9pm.

Grading Components

- Homework (30%): one assignment per week. Lowest one HW scores will be dropped.
- Midterm exam (30%): The midterm exam will be 80 minutes.
- Final (40%): You are allowed to finish the final exam in 2 hours.
- Please check the syllabus for more details.

What is probability

Outline

- Topics:
  - Definitions of Probability, Sample Space and Events
  - Probability Rules (The Complement Rule, General Addition Rule, Partition Rule, Containment Rule)
- Reading: Syllabus (available for download on canvas), Section 3.1 of OIS4ed and Section 2.1 - 2.3 of MMSA.
- Download R or RStudio and complete Lab0 & Lab1 (link provided on canvas).

What is probability

A model is a theoretical explanation of the phenomenon under study (experiment).

- Deterministic model: candidates of the experiments determine exactly one outcome. E.x. Drive with $v = 60$mph for $t = 0.5$ hour and observe the distance
- Stochastic experiments and models: the candidates of the experiments determine a collection of outcomes (outcome space), and a measure of likelihood (probability).

  E.x. Toss a fair coin and observe the uppermost face.

How are probabilities assigned?

Four approaches to define probability:

- classical approach;
- empirical approach;
- subjective approach;
- axiomatic approach.

An equally-likely assignment:

- Limitation: restricted only to a finite number of equally likely outcomes.
- E.x. Shuffle a deck of 52 cards and draw one at random. What is the probability to get the Ace of hearts? How about getting an Ace?

Deck of cards

A standard deck of cards has fifty-two playing cards.

- Thirteen ranks: 1 - 10, J, Q, K.
- Four French suits: clubs, spades, hearts, diamonds. Face cards: J, Q, K.
- $P(\text{Ace of hearts}) = ?$ $\quad P(\text{Ace}) = ?$

Empirical Probability

Based on repeated experimentation and observation:

- E.x. What is the probability that a breakthrough infection for people fully vaccinated against covid?

Law of Large Numbers

This illustrates the Law of Large Numbers (LLN), a theorem describing the result of performing the same experiment a large number of times.

Law of Large Numbers (LLN)

The average of the results obtained from a large number of trials should be close to the expected value, and will tend to become closer as more trials are performed.

- LLN guarantees stable long-term results for the averages of some random events.

Law of Large Numbers: Casino Example

For example, consider the game of the roulette wheel in a casino

- May earn/lose money in a single spin. Over a large number of spins, Earnings will tend towards a predictable percentage.
- The LLN only applies when a large number of observations is considered.
- For a small number of observations, there is no principle to guarantee to converge to the long term average.

Limitations

Three limitations:

- Impossible to have $N = \infty$ in practice.
- The probability assignments will be different for different experiments.
- Do not know how large N should be.

Subjective Probability and Axiomatic Definition

Subjective probability assigns probability based on a personal evaluation.

- important in decision making processes and Bayesian decision theoretical framework.

Axiomatic definition of probability defines probability based on a set of axioms that state the minimal requirements for probability.

- simplest and least controversial;
- compatible with the other definitions;
- allows for development of mathematical theorems;
- will be discussed in more advanced level courses.

What does probability mean?

However probability is defined, it should satisfy the following conditions.

- Probability is a number between 0 and 1.
- Probability near 1 indicates that the event is likely to occur.
- Probability near 0 indicates that the event is not likely to occur.
- Probability near $\frac{1}{2}$ indicates that the event is just as likely to occur as not.
- Probability does not mean whether or not the event will occur. Only the chance of occurrence.

Sample space and event

Previously discussed probability as a measure of likelihood for the candidates of the experiments.

Sample Space

The sample space S of a random phenomenon is the set of all possible outcomes.

Event

An event is a set of outcomes of a random phenomenon. That is, an event is a subset of the sample space. We will denote the set of all events by $\mathcal{S}$.

Sample Spaces and Events: Examples

Identify sample space and events in the following examples.

- Roll a fair six-sided die once.
- Flip a fair coin twice and observe the sequence of heads and tails.
- Suppose in above you measure the number of heads instead.

Overview of the set theory

Set is "a collection of definite, well distinguished objects of our perception or of our thought". (Georg Cantor, 1845-1918) Some important sets:

- Natural numbers: $N = \{1, 2, 3, \dots\}$
- Integers: $Z = \{\dots - 2, 1, 0, 1, 2, \dots\}$
- Real numbers: $R = (-\infty, \infty)$
- Intervals are denoted as follows:
  - $[0, 1]$: the interval from 0 to 1 including 0 and 1
  - $[0, 1)$: the interval from 0 to 1 including 0 but not 1
  - $(0, 1)$: the interval from 0 to 1 including neither 0 nor 1.
- If a is an element of the set A, then we write $a \in A$.
- If a is not an element of the set A, then we write $a \notin A$.

Sets denoted in Venn Diagrams

- S = Sample Space = collections of all the outcomes that can happen
- A = an event = a subset of S,
  B = another event = another subset of S.
- Denoted $A \subseteq S$ and $B \subseteq S$

Empty Set

- Empty set: set with no elements is called.
- Denoted $\emptyset$.
- For any set A, we have $\emptyset \subseteq A$.

Complement

- Set complement: set of all elements that are not in A.
- Denoted as A^c, read as A complement.

Intersection

- The intersection of A and B: set of all elements that are in both A and B.
- Denoted $A \cap B$.

Set Difference

- The set difference of A and B: set of all elements in A but not in B. Denoted $A \backslash B = A \cap B^c$.

Union

The union of A and B : Set of all elements that are either in A or in B or in both. Denoted $A \cup B$.

De Morgan's Laws

De Morgan's Laws

- $(A \cup B)^c = A^c \cap B^c$
- $(A \cap B)^c = A^c \cup B^c$

Practice: Set Operations

- What is $A \cup A^c, A \cap A^c, (A^c)^c, \emptyset^c$?
- What is S^c, where S is the sample space?

Disjoint (Mutually exclusive) Events

A and B are **disjoint** or **mutually exclusive** if they have no common elements, that is $A \cap B = \emptyset$.

Probability rules

Complement Rule

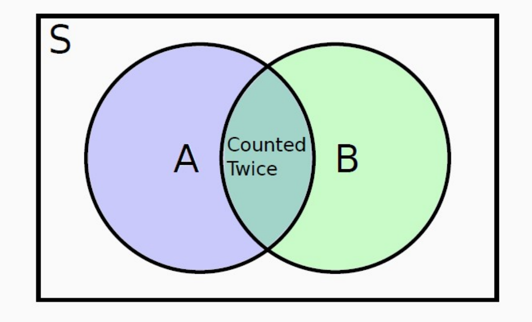
$$P(A^c) = 1 - P(A)$$

- Useful for understanding probability of events like $\{$less than or equal to $k\}$ since $\{$less than or equal to $k\}^c = \{$bigger than $k\}$.
- Ex. Rolling a pair of fair dice, what is the probability that the sum is at least 3?

Addition rule

General Addition Rule

$$P(A \cup B) = \begin{cases} P(A) + P(B) - P(A \cap B) & \text{in general,} \\ P(A) + P(B) & \text{if } A \cap B = \emptyset. \end{cases}$$



Ex. Rolling a pair of fair dice, what is the probability that the sum is 10?

- Ex. Rolling a pair of fair dice, what is the probability of getting a total of 10 or a double?

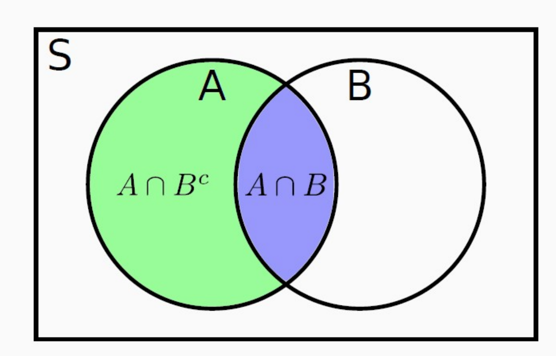
- what is the probability of getting a total of 10 or at least a 5?

Partition Rule

Let A and B be events in a sample space S.

Partition rule

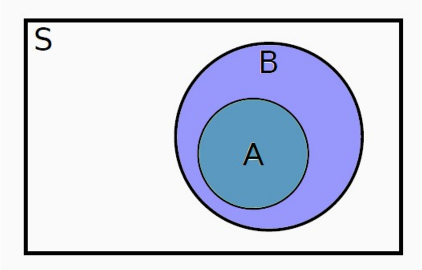
$$P(A) = P(A \cap B) + P(A \cap B^c)$$



Containment Rule

Containment rule

$P(A) \leq P(B)$ if $A \subseteq B$



Ex. In dice example, consider getting double ace v.s. any double

Linda problem

Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in antinuclear demonstrations.

Q: Which of the following is more probable?

1. Linda is a bank teller.
2. Linda is a bank teller and is active in the feminist movement.