Stat 23400 Lecture 2: Conditional probability and Bayes Theorem

Probability Rules: Review

- Probability and Events;
- Mutually exclusive (m.e.) events: Two events A and B are mutually exclusive (disjoint), if $A \cap B = \emptyset$.
- Probability Rules:
  1. The Complement Rule: $P(A^c) = 1 - P(A)$.
  2. General Addition Rule: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
  3. Addition Rule for m.e. events: $P(A \cup B) = P(A) + P(B)$ if A and B are m.e.
  4. Partition Rule: $P(A) = P(A \cap B) + P(A \cap B^c)$.
  5. Containment Rule: $P(A) \leq P(B)$ if $A \subseteq B$.

Outline

Topics: Conditional Probability, General Multiplication Rule; Independence; Bayes Theorem; Tree Diagram.

- Reading: Section 3.2, 3.3 in OIS and Section 2.4, 2.5 in MMSA.
- Assignment: Homework 1 on canvas.

Conditional Probability

Example: Poker

A card is drawn from a well-shuffled deck.

- What is the probability that the card drawn is a King?
- If the card drawn is known to be a face card (J, Q, K), what is the probability that it is a K?

Conditional probability

Conditional probability is a measure of the probability of an event occurring given that another event has occurred. The conditional probability of B given A is denoted as,

$$P(B|A).$$

For example, in the previous example knowing that the card is face card changed the sample space. As a result, we have

$$P(B|A) = \frac{4}{12} \neq P(B) = \frac{4}{52}.$$

Exercise

In the previous example, what is

- $P(A|B^c) = P(\text{ face card} \,|\, \text{not King })$?
- $P(A|B) = P(\text{ face card} \,|\, \text{King})$?
- $P(B|A^c) = P(\text{King} \,|\, \text{not face card})$?

Another Example

A deck of cards is well-shuffled and the two cards are drawn without replacement. What is the probability that second card is a King,

- given that the first card is a King?
- given that the first card is NOT a King?

Example (Deaths in the U.S. in 1996)

	Age						
Cause	1-4	5-14	15-24	25-44	45-64	≥ 65	All ages
Heart	207	341	920	16,261	102,510	612,886	733,125
Cancer	440	1,035	1,642	22,147	132,805	386,092	544,161
HIV	149	174	420	22,795	8,443	22	32,003
Accidents[1]	2,155	3,521	13,872	26,554	16,332	30,564	92,998
Homicide[2]	395	513	6,548	9,261	7,717	52	24,486
All causes	5,947	8,465	32,699	148,904	380,396	1,717,218	2,171,935

[1] Accidents and adverse effects, [2] Homicide and legal intervention

- P(accident) $= 92998/2171935 = 0.04282$
- $P(5 \leq age \leq 14) = 8465/2171935 = 0.00390$
- $P(\text{accident} \cap 5 \leq \text{ age } \leq 14) = 3521/2171935 = 0.00162$
- $P(\text{ accident}|5 \leq age \leq 14) = 3521/8465 = 0.41595$
- $P(\text{ accident}|25 \leq age \leq 44) = 0.17833$

Conditional Probability: Definition

Note that

$$P(\text{accident}|5 \le age \le 14) = \frac{3521/2171935}{8465/2171935}$$
$$= \frac{P(\text{accident and } 5 \le \text{age} \le 14)}{P(5 \le \text{age} \le 14)}$$

In general, we define conditional probability as follows.

Conditional Probability

The conditional probability $P(B|A)$ is related to $P(B \cap A)$ and $P(A)$ as follows.

$$P(B|A) = \frac{P(B \cap A)}{P(A)}, \qquad \text{if } P(A) > 0.$$

$$\boxed{P(B|A) = \frac{P(B \cap A)}{P(A)}, \qquad \text{if } P(A) > 0.}$$

Interpretation: Given A, the possible outcomes is restricted to A, Thus renormalize the probability such that $P(A|A) = 1$.

This identity formally defines conditional probability. In practice, we may calculate $P(B|A)$ directly by thinking about how A changed the sample space.

General Multiplication Rule

General Multiplication Rule

Recall the definition of conditional probability
$P(A \cap B) = P(A)P(B|A)$

- Multiplying by $P(A)$ on both sides $\Rightarrow$ **General Multiplication Rule**,

$$P(A \cap B) = P(A) \times P(B|A)$$

- Knowing $P(A), P(B|A)$, $P(A \cap B)$ can be calculated using the General Multiplication Rule.
- General multiplication Rule for several events:

$$\begin{aligned} P(\cap_{i=1}^{n} A_i) &= P(A_1 \cap A_2 \cdots \cap A_n) \\ &= P(A_1) \times P(A_2|A_1) \times \cdots P(A_n|A_{n-1} \cap \cdots \cap A_1) \end{aligned}$$

A deck of cards is shuffled and the two top cards are placed face down on a table.

- What is the probability that at least one card is a King?

- Given that the 1st card is not a K, the conditional probability that the 2nd card is not a $K = ?$

- The probability that neither card is a $K =?$

- $P(\text{at least one of the two cards is a } K) =?$

Independence

Independence

Two events A and B are said to be independent, if and only if

$$P(A \cap B) = P(A) \times P(B)$$

A and B are independent, if any of the following is true

- $P(A \cap B) = P(A) \times P(B)$ (by definition)
- $P(A|B) = P(A)$ (B happens doesn't affect how likely A happens)
- $P(A|B) = P(A|B^c)$
  (How likely A happens is not affected by B happens or not)
- $P(B|A) = P(B)$ (A happens doesn't affect how likely B happens)

Some Proofs

- Proof of $P(A|B) = P(A)$ implies $P(A \cap B) = P(A)P(B)$
- Proof of $P(A|B) = P(A)$ implies $P(B|A) = P(B)$
- Proof of $P(B|A) = P(B)$ implies $P(A \cap B) = P(A)P(B)$.

Example

- If one rolls a fair die twice, what is probability 2nd roll shows 1?
- what is probability 2nd roll shows 1 if 1st roll showed 1?
- . . . and if the 1st roll showed 2?
- . . . and if the 1st roll showed whatever?

Chance of 2nd roll showing 1 stays the same, no matter what the 1st shows. Thus, they are independent.

Independence vs Disjoint Events

- If A and B are independent, $P(A \cap B) = P(A) \times P(B)$.
- If A and B are disjoint: $A \cap B = \emptyset \Rightarrow P(A \cap B) = 0$.
- If $P(A) > 0$ and $P(B) > 0$,
  - Independent events cannot be disjoint.
  - Disjoint events cannot be independent.
- Conceptually, A and B are disjoint means that one happens prevents the other from happening, so one's occurrence definitely affects the other's.

E.x. Two chips are selected at random with replacement from a bowl with 4 red and 6 green. What is the probability of getting a red in the second draw if we know that the first draw is a red? What is the probability that both are with the same color?

Independent Events: Example (2)

What if we draw the chips without replacement?

Bayes Theorem and Tree Diagrams

A Nervous Job Applicant

Suppose an job applicant has been invited for an interview. The probability that

- he is nervous is $P(N) = 0.7$,
- the interview is successful given he is nervous is $P(S|N) = 0.2$,
- the interview is successful given he is not nervous is $P(S|N^c) = 0.9$.

Questions of interest:

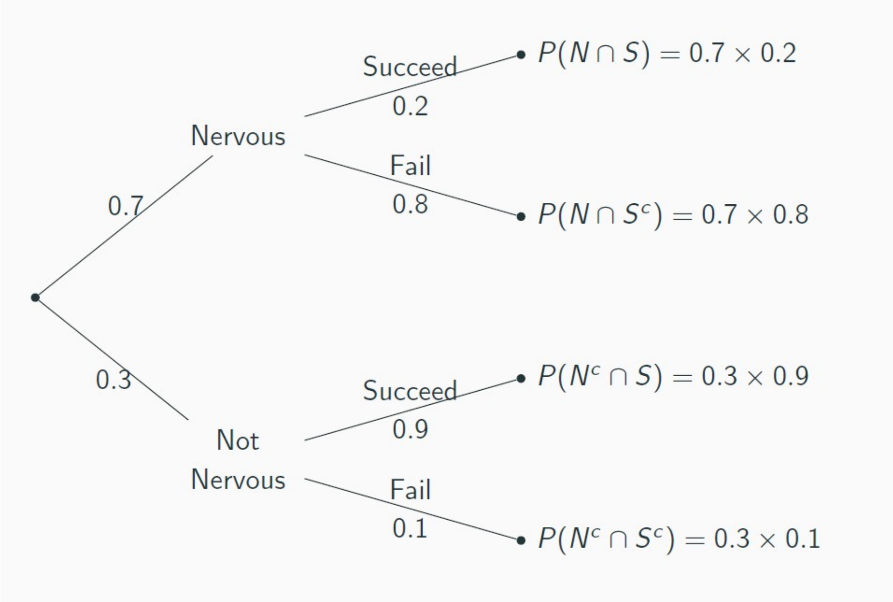
- Q1: What is the probability that the interview is successful?
- Q2: Given the interview is successful, what is the probability that the job applicant is nervous during the interview?

Solution: Nervous Job Applicant Example

- Q1: What is the probability that the interview is successful?

- Q2: Given the interview is successful, what is the probability that the job applicant is nervous during the interview?

Tree Diagram for the Nervous Job Applicant Example



Bayes' Theorem

The problem on the previous slide is an example of the Bayes' Theorem.

Bayes' Theorem

Let A and B be two events. We have

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

- Proof: by the def of conditional probability and partition rule of probability.
- In order to calculate $P(A|B)$, in addition to $P(B|A)$, we need to know $P(B|A^c)$ and $P(A)$ as well.

Proof of Bayes' Theorem

Bayes' Theorem in Medical Testing

Bayes' Theorem is very useful in medical testing.

- D = the event that the individual has the disease we are testing for
- Let T_+ denote the event that the test result is positive, and T_- denote the event that the test comes back negative
- $P(T_+|D)$ is called the sensitivity of the test
- $P(T_-|D^c)$ is called the specificity of the test
- Ideally, both $P(T_+|D)$ and $P(T_-|D^c)$ would equal 1. However, diagnostic tests are not perfect. They may give false positives and false negatives.
- What we really want to know are $P(D|T_+)$ and $P(D|T_-)$.

Example: Mammogram Screening for Breast Cancer (OpenIntro, p.105)

If we know that

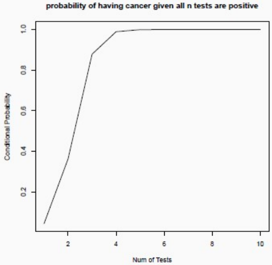
- $P(T_+|D) = 0.89$ (sensitivity - positive for disease)
  $P(T_-|D^c) = 0.93$ (specificity - negative for no disease)
- In Canada, about 0.35% of women over 40 will develop breast cancer in any given year, a.k.a., $P(D) = 0.0035$.

What is the probability that a tested Canadian woman has breast cancer if the test was positive?

- Even if the mammogram comes back positive, there is still only a 4% chance that she has breast cancer. Need to confirm by a second test.
- What is the probability that a tested woman has breast cancer if she was tested independently for n times and all the tests were positive?

```r
CalProbD <- function( n, pd, psen, pspec ) {
     result <- pd * psen^n / ( pd * psen ^ n + ( 1 - pd ) * ( 1 - pspec )
^ n )
     result
}

ConP <- CalProbD(1:10, 0.0035, 0.89, 0.93)
```



Bayes' Theorem for 3 or More Cases

- In the previous two examples, sample space were only split into two parts B or B^c (nervous or not nervous, cancer or no cancer).
- In some cases, we may need to calculate $P(A)$ by splitting it into several parts, using the law of total probability:
  Suppose $B_1 \cup B_2 \cup \cdots \cup B_K = S$ and $B_i \cap B_j = \emptyset$ for all $i \neq j$, then

$$
\begin{aligned}
P(A) &= P(A \cap B_1) + P(A \cap B_2) + \cdots + P(A \cap B_K) \\
&= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \cdots + P(A \cap B_K)P(B_K).
\end{aligned}
$$

- Using law of total probability, the Bayes Theorem becomes

$$
P(B_i|A) = \frac{P(A|B_i)P(B_i)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \cdots + P(A|B_K)P(B_K)}
$$

Example: Terrorism and Party Identification

In a certain city

- 40% of the registered voters are Democrats,
- 32% are Republicans, and
- 28% are Independents.

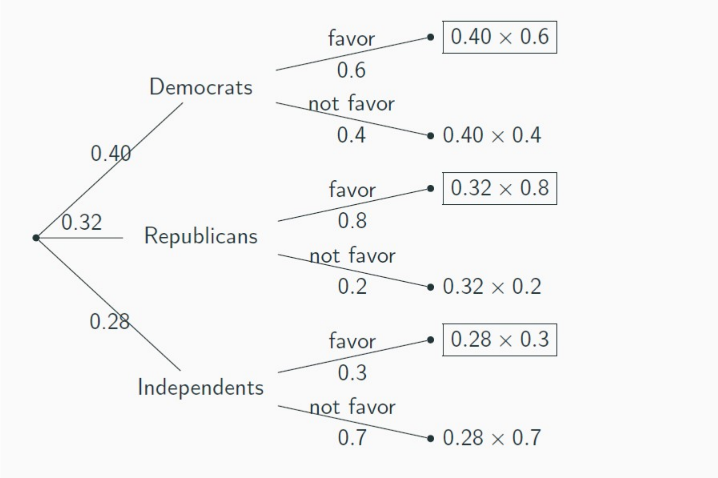
Moreover,

60% of the Democrats, 80% of the Republicans, and 30% of the Independents favor increased spending to combat terrorism.

Q1: What percentage of the registered voters in this city favor increased spending to combat terrorism? P(favor) $=$?

Q2: If a person chosen at random from this city favors increased spending to combat terrorism, what is the probability that he/she is a Republican? $P(R|$ favor$) =$?

Terrorism and Party Identification Example: Tree Diagram



Q1: What percentage of the registered voters in this city favor increased spending to combat terrorism?

Q2: If a person chosen at random from this city favors increased spending to combat terrorism, what is the probability that he/she is a Republican?