STAT 23400 Lecture 3: Discrete random variable and expected value

Review

- Conditional probability: $P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(BA)}{P(A)}$
- Independence: $P(AB) = P(A)P(B)$
- Bayes Theorem and Tree Diagram. Let A and B be two events, then

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

- Bayes Theorem for 3 or more cases.

Outline for Today

- Topics: Random variables, the probability mass function (pmf) for a discrete r.v., expected value.
- Reading: Section 3.4, 4.2 (Geometric) in OIS and Section 3.1-3.3 in MMSA.
- Lab02: Introduction to R Markdown. Link posted on canvas.

Random Variable

Random variable

A random variable is a real-valued function on the sample space S and maps elements of S, ω, to real numbers.

$$S \xrightarrow{X} \mathbb{R}$$
$$\omega \to x = X(\omega)$$

Two types of r.v.'s:

- Discrete random variables: can only take a finite or countably infinite number of different values # of students enrolled in Stat234.

- Continuous random variables: take real (decimal) values, e.g., time spent working on Stat234 last week.

Distribution of a Discrete Random Variable

The probability distribution or the distribution of a discrete random variable X is a list of the probability p_i for x_i , where x_i are the possible values it may take:

Value of X	x_1	x_2	x_3
Probability	p_1	p_2	p_3

The probabilities p_i must satisfy two requirements: $p_i \geq 0$ and

$$\sum_i p_i = 1.$$

Coin Example

Let X be the number of heads in 4 tosses of a fair coin. What is the probability distribution for X?

Probability Mass Function

For a discrete random variable X with $x_1, \ldots, x_n, \ldots$ being its possible values, the probability mass function (pmf) is a function $p(x)$ that maps x_i to the corresponding probability $P(X = x_i)$, i.e.,

$$\mathbb{R} \xrightarrow{p} [0,1]$$
$$x \longrightarrow p(x)$$

Example: Coin Tossing

Recall that in the coin tossing example, X is the number of heads in 4 tosses of a fair coin, and

Possible Values of X	0	1	2	3	4
Probabilities	$1/16$	$4/16$	$6/16$	$4/16$	$1/16$

What is the pmf of X?

Example: Geometric Distribution

Let X be the number of tosses needed to obtain the first heads, when tossing a coin with a probability of p to land heads.

Find the pmf.

Example: Geometric Distribution

- We say X has a geometric distribution since the pmf is a geometric sequence.

- The geometric probability models $n-1$ failures before the first success, or e.g., total number of children until the first girl.

- Does $\sum_{x=1}^{\infty} p(x) = 1$? (This will be your homework.)

Review: Summation Notation and Its Properties

Consider a sequence $\{x_i, i = 1, \ldots n\}$, we have

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$$

In the following, a is a fixed constant.

$$\sum_{i=1}^{n} a = \underbrace{a + a + \ldots a}_{n \text{ copies}} = na$$

$$\sum_{i=1}^{n} (ax_i) = ax_1 + ax_2 + \cdots + ax_n$$

$$= a(x_1 + x_2 + \cdots + x_n) = a \sum_{i=1}^{n} x_i$$

$$\sum_{i=1}^{n}(x_i + y_i) = (x_1 + y_1) + (x_2 + y_2) + \cdots + (x_n + y_n)$$

$$= (x_1 + x_2 + \cdots + x_n) + (y_1 + y_2 + \cdots + y_n)$$

$$= \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i$$

Expected Value of a Discrete R.V.

Expected Value of a Discrete R.V.

Let X be a discrete random variable with pmf p(x). The expected value of X, denoted by $E[X]$, is defined as

$$E[X] = \sum_{\text{all } x} xP(X = x)$$
$$= \sum_{\text{all } x} xp(x)$$

provided that $\sum_x |x|p(x) < \infty$.

$$\boxed{E(X) = \sum_{\text{all } x} xp(x)}$$

The expected value is a weighted average of the possible values of X, where the weights are the probabilities of those values.

- If $\sum_x |x|p(x) = \infty$, the expected value does not exist.
- The expected value is also called expectation or mean, and sometimes we use $\mu = E[X]$ to represent the expected value.

Example: Card game

Here is the rule of a card game. You pick a card from a deck. If the card that you pick is king of spades, then you earn 10 dollars. If it is Ace, then you earn 5 dollars. If it is a heart (but not Ace), then you earn 1 dollar.

Example: Card game

The Card Game example.

$$p(x) = \begin{cases} 35/52 & \text{if } x = 0 \\ 12/52 & \text{if } x = 1 \\ 4/52 & \text{if } x = 5 \\ 1/52 & \text{if } x = 10 \\ 0 & \text{else} \end{cases}$$

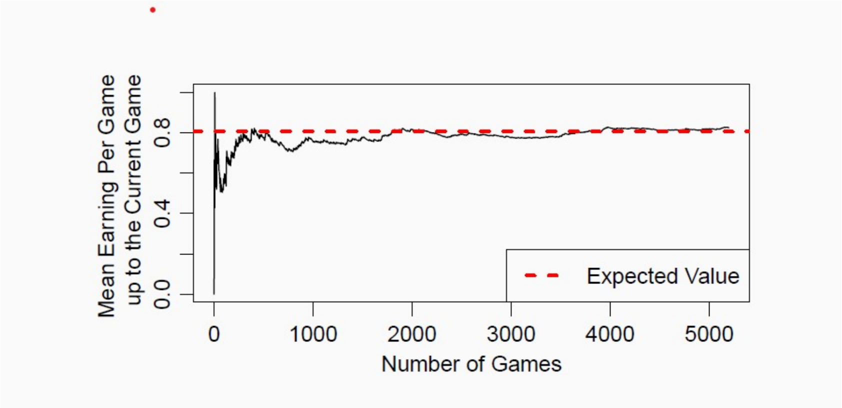
Find the expected reward.

Expectation: Interpretation

The expected value is the long run average earning in a game.

- Ex. Suppose someone plays the card game 5200 times. Each time the card is drawn with replacement. What is the average reward?

Expectation is the Long Term Average



So $E(X)$ is average of X when we repeat a HUGE number of times.

Fair game

A fair game is defined as a game that costs as much as its expected payout, i.e. expected profit is 0.

For the card game we have discussed so far,

- will you play the game if it costs 1 to play once?
- will you play the game if it costs 50 cents to play once?
- what is the maximum amount you would be willing to pay to play this game?

Find the expected value for the coin tossing example, as follows

Possible values of X	0	1	2	3	4
Possibilities	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

Exercise 2: Geometric distribution

Find the expected value for the Geometric Distribution,

$$p(x) = (1-p)^{x-1}p \qquad x = 1, 2, 3, \ldots$$

Function of a random variable

Function of a Random Variable: An Example

For $X = \#$ of heads obtained in 4 tosses of a fair coin, recall the pmf of X is as follows.

Possible values of X	0	1	2	3	4
Possibilities	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

Suppose for some reason, we are interested in $Y = (X - 1)^2$, which is a function of X.

- Note that Y is a random variable, too.
- What are the possible values of Y ?
- What is the distribution of Y?

Find the distribution of Y.

possible values of X	pmf of X $p_X(x)$	value of $Y = (X-1)^2$
0	1/16	$(0-1)^2 = 1$
1	4/16	$(1-1)^2 = 0$
2	6/16	$(2-1)^2 = 1$
3	4/16	$(3-1)^2 = 4$
4	1/16	$(4-1)^2 = 9$

What is the expected value of $Y = (X - 1)^2$?

Discussion

In this example, to find $E[Y]$, we have to

- first find the pmf $p(y)$ of $Y = g(X)$, which may not be easy if $g()$ is a complicated function, and
- then calculate $E(g(X))$ as $\sum_y y p_Y(y)$ using the pmf of Y.
- There is a much easier way to find $E[g(X)]$ without first finding the pmf of $Y = g(X)$. See the next slide.

Function of a Discrete R.V. Expected Value

Expected Value of a Function of a Discrete R.V.

If the pmf of X is $p(x)$, the expected value of $g(X)$ is

$$E[g(X)] = \sum_x g(x)p(x);$$

provided that $\sum_x |g(x)|p(x) < \infty$.

Example: Card Game

Recall the pmf of the reward of the card game is

x	0	1	5	10
$p(x)$	35/52	12/52	4/52	1/52

If it costs 1 dollar to play the game, find the expected net profit.