STAT 23400 Lecture 4: Variance, Linear transformation and Moment generating function

Review

- Expected Value of Discrete R.V.'s:

$$\mu = E[X] = \sum_{\text{all } x} x p(x), \text{ if } \sum_x |x| p(x) < \infty.$$

- Expected Value of Functions of Discrete R.V.'s:

$$E[g(X)] = \sum_x g(x) p_X(x) \text{ if } \sum_x |g(x)| p_X(x) < \infty.$$

Outline

Topics: variance, linear transformation of a r.v., moments, and moment generating function.

- Reading: Section 3.4 in OIS and Section 3.3 - 3.4 in MMSA.
- Assignment: HW2 on canvas.

Variance of a random variable

Variance: Introduction

The expected value $E(X)$ tells us the center of the probability distribution for a r.v.

Sometimes we are interested in the variability or dispersion.

- Does the expected value tell us how dispersed the distribution is?
- Can we use $E(X - \mu)$ to measure dispersion?

Variance of a random variable

The variance of a random variable X, denoted as $Var(X)$ is defined as

$$Var(X) = \sigma^2 = \text{“sigma squared”} = E((X - \mu)^2)$$

- Variance is the average squared distance from the mean.
- Variance is a measure of how a random variable is deviated from its mean, a.k.a. spread of its probability distribution.
- Take the square root to determine the standard deviation (SD).

$$SD(X) = \sigma = \sqrt{Var(X)}$$

Example: Card game

In the example of the reward X while playing the card game, recall that $\mu = E(X) = 42/52$ and the pmf of X is

x	0	1	5	10
$p(x)$	35/52	12/52	4/52	1/52

What is the variance of X?

A computational formula for $Var(X)$ is

$$Var(X) = E((X - \mu)^2) = E(X^2) - \mu^2$$

Example: Card Game

Recall the card game. The pmf is

x	0	1	5	10
$p(x)$	$35/52$	$12/52$	$4/52$	$1/52$

Calculate the variance again using the computational formula.

Linear Transformation of a Random Variable

Linear Transformation of a Random Variable

Linear transformation of a random variable $g(X) = aX + b$ is also a function that is often of interest.

- For example, consider the change of unit: $F = 32 + \frac{9}{5}C$ for temperature or conversion between the English and Metric units.
- How do the expected value and variance change under linear transformation?

Expected Value Under Linear Transformation

Let X be a discrete r.v., with pmf p(x). We have

$$E(aX + b) = aE(X) + b.$$

Variance under Linear Transformation

Let Y = aX + b, we can show that

$$Var(Y) = a^2 Var(X)$$

Moments of a random variable

Moments of a Random Variable

We define the kth moment of X, $k = 1, 2, 3, \ldots$ as

$$E(X^k) = \sum_x x^k p(x)$$

provided that $\sum_x |x|^k p(x) < \infty$.

- The first moment $E(X)$ is simply the mean of X.
- $E[(X-\mu)^k]$ is called the kth moment about the mean.
- The second moment about the mean $E(X-\mu)^2$ is the variance of X.

$$\begin{aligned} Var(X) &= E[(X-\mu)^2] \\ &= (\text{second moment}) - (\text{square of the first moment}) \end{aligned}$$

Example: Coin Tossing, 3rd Moment

Recall the pmf for $X = \#$ of heads obtained in 4 tosses of a fair coin is

x	0	1	2	3	4
$p(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

Previously, we have calculated the mean is 2.

Q. Find the third moment about the mean.

Example: Card Game, 3rd Moment

Recall the pmf of the reward X of playing the card game once is

x	0	1	5	10
$p(x)$	35/52	12/52	4/52	1/52

Q. Find the third moment about the mean. Is the pmf symmetric about μ?

Moment generating function

Moment Generating Functions

For a discrete r.v. with pmf p(x), the moment generating function (mgf) is defined as

$$M_X(t) = E(e^{tX}) = \sum_x e^{tx} p(x), \quad t \in R.$$

- Mgf is the expected value of $g(X) = e^{tX}$.
- Mgf can be used to find the moments of the distribution (next slide).
- Mgf is a function about t, instead of x.

Recall that

$$\frac{d}{dt}e^{tx} = xe^{tx}$$

We can show the following facts are true (proofs at the end of this lecture)

- $M_X(0) = E[e^{0 \cdot X}] = E[e^0] = E[1] = 1.$
- $M'_X(t) = E[Xe^{tX}]$ and $M'_X(0) = E[X]$.
- $M_X^{(2)}(t) = E[X^2 e^{tX}]$ and $M_X^{(2)}(0) = E[X^2]$.
- In general, one can show that

$$M_X^{(k)}(t) = E[X^k e^{tX}] \text{ and hence } M_X^{(k)}(0) = E[X^k].$$

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Example: Card Games

Recall the pmf of the reward X of playing the card game once is

x	0	1	5	10
$p(x)$	$35/52$	$12/52$	$4/52$	$1/52$

Find the moment generating function for X.

Recall the pmf for the Geometric distribution is

$$p(x) = P(X = x) = (1 - p)^{x-1} p, \qquad \text{for } x = 1, 2, 3, \ldots$$

Find the mgf for the Geometric distribution.

Example: Finding the Expected Value Using the MGF

Recall the mgf for the Geometric distribution

$$M(t) = \sum_{x=1}^{\infty} e^{tx}(1-p)^{x-1}p = \frac{pe^t}{1-e^t(1-p)} \qquad \text{if } e^t(1-p) < 1$$

Find the expected value and variance using mgf.

$$\mu = \mathbb{E}[X] = M'_X(0) = \boxed{\frac{1}{p}}$$

$$\mathbb{E}[X^2] = M''_X(0) = \frac{2}{p^2} - \frac{1}{p}$$

$$\text{Var}[X] = \mathbb{E}[X^2] - \mu^2 = \boxed{\frac{1}{p^2} - \frac{1}{p}}$$

Proofs of the Properties of MGF

Proofs: Properties of MGF

- $M_X(0) = E[e^{0 \cdot X}] = E[e^0] = E[1] = 1$
- $M'_X(t) = E[Xe^{tX}]$
  Proof.

$$
\begin{aligned}
M'_X(t) = \frac{d}{dt} M_X(t) = \frac{d}{dt} \sum_x e^{tx} p(x) \\
= \sum_x \frac{d}{dt} e^{tx} p(x) \\
= \sum_x x e^{tx} p(x) = E[Xe^{tX}]
\end{aligned}
$$

- Hence $M'_X(0) = E[Xe^{0 \cdot X}] = E[X]$.

Second Derivative of the Moment Generating Function

$$M_X^{(2)}(t) = E[X^2 e^{tX}] \text{ and hence } M_X^{(2)}(0) = E[X^2]$$

Proof.

$$\begin{aligned}
M_X^{(2)}(t) = \frac{d}{dt} M_X'(t) &= \frac{d}{dt} E[Xe^{tX}] \\
&= \frac{d}{dt} \sum_x x e^{tx} p(x) \\
&= \sum_x \frac{d}{dt} x e^{tx} p(x) \\
&= \sum_x x^2 e^{tx} p(x) = E[X^2 e^{tX}]
\end{aligned}$$

Hence, $M_X^{(2)}(0) = E[X^2 e^{0X}] = E[X^2]$.

kth Derivative of the Moment Generating Functions

In general, one can show that $M_X^{(k)}(t) = E[X^k e^{tX}]$ and hence $M_X^{(k)}(0) = E[X^k]$

- The kth derivative of mgf evaluated at $t = 0$ is just the kth moment of the random variable
- This why $M_X(t) = E[e^{tX}]$ is called the **moment generating function**.
- Mgfs have more useful properties that we will see the power later.