STAT 23400 Lecture 8: Joint Probability Distribution for Discrete and Continuous Random Variables

Review: Random Variables

- Random Variable: a mapping from sample space to real values. A r.v. is discrete if it can only take countable values.

- Discrete Random Variable: Probability Mass Function (pmf), Expected Value, Variance, SD, Linear Transformation, Moments and Moment Generating Function (mgf).

- Continuous Random Variable: Probability Density Function (pdf), Cumulative Distribution Function (cdf), Expected Value, Variance, SD, Linear Transformation, Moments and Moment Generating Function (mgf).

- Special Cases: Binomial Distribution and Normal Distribution.

Outline

Topics:

- Joint Probability Distributions (Section 5.1 in MMSA).
- Conditional Distributions and Independence (Section 5.3 in MMSA).
- Assignment: Homework 4 on canvas.

Joint Probability Distribution: Marginal distribution and Conditional distribution

The pmf of one of the variables alone is obtained by summing $p(x, y)$ over values of the other variable, called a **marginal pmf**. For example, the marginal pmf of X can be obtained as

$$p_X(x) = P(X = x) = \sum_y P(\{X = x\} \cap \{Y = y\}) = \sum_y p(x, y).$$

and the marginal pmf of Y can be obtained as

$$p_Y(y) = P(Y = y) = \sum_x P(\{X = x\} \cap \{Y = y\}) = \sum_x p(x, y).$$

The marginal distribution of X is the row sum of the joint pmf.

	$p(x,y)$	y: 1	2	3	4	5	6	Row Sum $p_X(x)$
x	1	1/36	2/36	2/36	2/36	2/36	2/36	11/36
	2	0	1/36	2/36	2/36	2/36	2/36	9/36
	3	0	0	1/36	2/36	2/36	2/36	7/36
	4	0	0	0	1/36	2/36	2/36	5/36
	5	0	0	0	0	1/36	2/36	3/36
	6	0	0	0	0	0	1/36	1/36
Column Sum	$p_Y(y)$	1/36	3/36	5/36	7/36	9/36	11/36	1

Conditional Distributions of Y given $X = x$

The **conditional pmf** of a discrete r.v. Y given another discrete r.v. $X = x$ is

$$p_{Y|X=x}(y|x) = \frac{P(X = x, Y = y)}{P(X = x)} = \frac{p(x, y)}{p_X(x)}.$$

in which $p(x, y)$ is the joint pmf of X and Y, and $p_X(x)$ is the marginal pmf of X.

Given $X = 3$, what is the probability that $Y = 4$?

	$p(x,y)$	1	2	3	4	5	6	$p_X(x)$
	1	1/36	2/36	2/36	2/36	2/36	2/36	11/36
	2	0	1/36	2/36	2/36	1/36	2/36	9/36
x	3	0	0	1/36	2/36	2/36	2/36	7/36
	4	0	0	0	1/36	2/36	2/36	5/36
	5	0	0	0	0	1/36	2/36	3/36
	6	0	0	0	0	0	1/36	1/36

Find the conditional pmf of Y given X $= 5$

	$p(x,y)$	1	2	3	4	5	6	$p_X(x)$
	1	1/36	2/36	2/36	2/36	2/36	2/36	11/36
	2	0	1/36	2/36	2/36	1/36	2/36	9/36
x	3	0	0	1/36	2/36	2/36	2/36	7/36
	4	0	0	0	1/36	2/36	2/36	5/36
	5	0	0	0	0	1/36	2/36	3/36
	6	0	0	0	0	0	1/36	1/36

$p_{Y\|X}(y\|x)$		y = bigger number 1	2	3	4	5	6	Row Sum
	1	1/11	2/11	2/11	2/11	2/11	2/11	1
	2	0	1/9	2/9	2/9	2/9	2/9	1
x	3	0	0	1/7	2/7	2/7	2/7	1
	4	0	0	0	1/5	2/5	2/5	1
	5	0	0	0	0	1/3	2/3	1
	6	0	0	0	0	0	1	1

- Each row is a pmf for Y given some x value
- Observed the row sums of $p_{Y|X}(y|x)$ are all 1

Conditional Distributions of X given Y

The conditional pmf of a discrete r.v. X given another discrete r.v. $Y = y$ is

$$p_{X|Y}(x|y) = \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{p(x, y)}{p_Y(y)}.$$

in which $p(x, y)$ is the joint pmf of X and Y, and $p_Y(y)$ is the marginal pmf of Y.

	$p_{X\|Y}(x\|y)$	1	2	3	4	5	6
	1	1	2/3	2/5	2/7	2/9	2/11
	2	0	1/3	2/5	2/7	2/9	2/11
x	3	0	0	1/5	2/7	2/9	2/11
	4	0	0	0	1/7	2/9	2/11
	5	0	0	0	0	1/9	2/11
	6	0	0	0	0	0	1/11
Column Sum		1	1	1	1	1	1

- Each column is a pmf for X given some y value
- Observed the column sums $p_{X|Y}(x|y)$ are all 1

In summary, a conditional pmf of Y given X must satisfy

$$0 \le p_{Y|X}(y|x) \le 1 \quad \text{and} \quad \sum_y p_{Y|X}(y|x) = 1, \text{ for all } x.$$

A conditional pmf of X given Y must satisfy

$$0 \le p_{X|Y}(x|y) \le 1 \quad \text{and} \quad \sum_x p_{X|Y}(x|y) = 1, \text{ for all } y.$$

Conditional expectation

Using conditional pmf, we can define the conditional expectation as

$$E[Y|X = x] = \sum_{y} y p_{Y|X}(y|x)$$

and

$$E[X|Y = y] = \sum_{x} x p_{X|Y}(x|y)$$

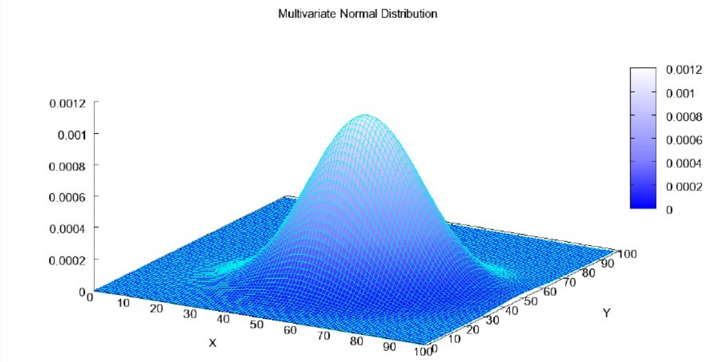
Two dice are rolled. The numbers on the top face of the two dice are recorded. Let X be **the smaller number**, and Y **the bigger one**. Questions of interest include:

- What is the distribution (pmf) of X? of Y?
- $E(X) = ?E(Y) = ?$
- Suppose you are told that $X = 3$
  - What are the possible values of Y ?
  - Is Y equally likely to be 3, 4, 5, or 6?
  - Given $X = 3$, what is the distribution of Y?
  - Given $X = 3$, what is $E(Y|X = 3)$?

Joint Distribution for Two Continuous R.V.s

Def. Let X and Y be two continuous r.v.'s. $f(x, y)$, the joint pdf for X and Y is defined such that for any set A,

$$P((X, Y) \in A) = \int\int_A f(x, y) dxdy.$$


Multivariate Normal Distribution
0.0012
0.001
0.0008
0.0006
0.0004
0.0002
0
X
Y
0
10
20
30
40
50
60
70
80
90
100

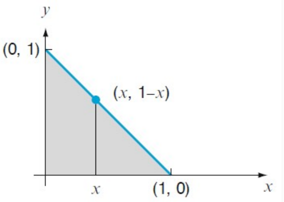
Example 5.5 on p.237-238 of MMSA

A can of mixed nuts contains almonds, cashews and peanuts. The weight of each can is 1 lb. Let X be the weight of almonds in a selected can and Y be the weight of cashews. The joint pdf for X and Y is given as

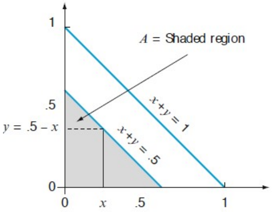
$$f(x, y) = \begin{cases} 24xy & \text{if } 0 \leq x \leq 1, 0 \leq y \leq 1, x + y \leq 1, \\ 0 & \text{elsewhere .} \end{cases}$$

- Is $f(x, y)$ a joint pdf?
- What is the probability that the two types of nuts together make up at most 50% of the can?

Example 5.5 Part 1



Example 5.5 Part 2



Marginal Density

Def. Given two continuous random variables with the joint pdf $f(x, y)$, the **marginal probability density function**, or simply marginal density of X and Y, are defined as

$$f_X(x) = \int_{\mathbb{R}} f(x, y)dy \text{ for all } x \in \mathbb{R}$$
$$f_Y(y) = \int_{\mathbb{R}} f(x, y)dx \text{ for all } y \in \mathbb{R}$$

Marginal Density: Example

Back to Example 5.5. Find the marginal pdf for X (almond) and Y(Cashew), respectively.

Def. Given two continuous random variables with the joint pdf $f(x, y)$, the conditional probability distribution of X given $Y \in dy$ is the function $f_{X|Y}$, and the conditional probability distribution of Y given $X \in dx$ is the function $f_{Y|X}$ are defined as

$$f_{X|Y=y}(x|y) = \frac{f(x,y)}{f_Y(y)}, \qquad f_{Y|X=x}(y|x) = \frac{f(x,y)}{f_X(x)},$$

Conditional Density: Example

In Example 5.5, how about the conditional pdf $f_{X|Y=y}(x|y)$ and $f_{Y|X=x}(y|x)$?

Independent R.V.

Independent Random Variables

Definition (Independent Random Variables): two random variables X and Y are independent if and only if

$$p(x, y) = p_X(x)p_Y(y) \qquad \text{if } X \text{ and } Y \text{ are discrete },$$
$$f(x, y) = f_X(x)f_Y(y) \qquad \text{if } X \text{ and } Y \text{ are continuous}$$

for all x and y.

Two use of independence:

- Check whether two r.v.'s are independent.
- Given that two r.v.'s are independent and their marginal distributions, we can find the joint distribution.

Example 1: Are X and Y Independent (Discrete Case)?

Suppose X and Y are discrete random variables, with $p(x, y)$ given as below. Are they independent?

$p(x, y)$	1	2	3
1	0.05	0.10	0.05
2	0.10	0.40	0.10
3	0.05	0.10	0.05

Example 2: Are They Independent? (Continuous Case)

Recall that in E.x. 5.5 (MMSA), X is the percentage of almond and Y is the percentage of cashew. Are they independent?

Example 3: Are They Independent? (Continuous Case)

Suppose the joint pdf of X, Y is

$$f(x, y) = 4xy, \text{ for } 0 \leq x, y \leq 1$$

Are they independent?

Finding Joint Distribution when Independent

If X and Y are independent, then

$$p(x, y) = p_X(x)p_Y(y) \text{ for discrete } X \text{ and } Y,$$
$$f(x, y) = f_X(x)f_Y(y)\text{for continuous } X \text{ and } Y.$$

Example: Finding Joint Distribution (Discrete Case)

Ex1. Suppose X and Y are independent random variables, with probability distributions

y	1	2	3
$p_Y(y)$	0.2	0.6	0.2

x	1	2	3
$p_X(x)$	0.2	0.6	0.2

Example: Finding Joint Distribution (Continuous Case)

If X and Y are independent with marginal pdfs

$$f_X(x) = e^{-x} \text{ and } f_Y(y) = 2e^{-2y},$$

for $0 < x, y < \infty$. Find their joint pdf.

A Simple Criterion for Checking Independence

So far, it seems like one at least has to find the marginal distribution first before checking for independence. However, there is an easier way...

A Simple Criterion

X and Y are independent if the joint pmf/pdf can be written as the product of a function of x and a function of y.

$$f(x, y) = g(x)h(y), \text{ for all } x, y.$$

Here, $g(x) \geq 0$ and $h(y) \geq 0$ **are not necessarily pmfs/pdfs.**

Exercise: Are They Independent?

- $p(x,y) = \frac{1}{18}xy$ for $x \in \{1,2\}$ and $y \in \{1,2,3\}$.
- $p(x,y) = (x+y)/36$ for $x, y \in \{1,2,3\}$.
- $f(x,y) = e^{-2}/x!y!$ for $x, y \in \{0,1,2\}$.
- $f(x,y) = 8xy$ for $0 \le x < y \le 1$.

- If we know the joint distribution/density of (X, Y), we can **always** derive the marginal distributions/densities of X and Y (and therefore the conditionals),

$$p_X(x) = \sum_y p(x,y), \qquad f_X(x) = \int_{\mathbb{R}} f(x,y)dy$$

- If we know the marginals and that $X$and Y are independent, we can derive the joint distribution/density of (X, Y)

$$p(x,y) = p_X(x)p_Y(y), \qquad f(x,y) = f_X(x)f_Y(y).$$

- If we know the marginal of X and the conditional of $Y|X$, we can derive the joint distribution/density of (X, Y) (By definition, independence is not required),

$$p(x,y) = p_X(x)p_{Y|X}(y|x) \qquad f(x,y) = f_X(x)f_{Y|X}(y|x).$$