Short time large deviations of the KPZ equation

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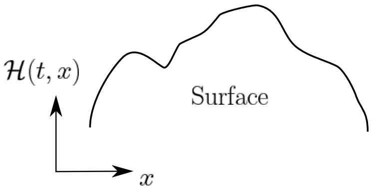
KPZ equation

- Introduced by Kardar-Parisi-Zhang (1986)

$$\partial_t \mathcal{H}(t,x) = \frac{1}{2}\partial_{xx}\mathcal{H}(t,x) + \frac{1}{2}\big(\partial_x \mathcal{H}(t,x)\big)^2 + \xi(t,x)$$

ξ is the space-time white noise.



Stochastic heat equation

- Stochastic heat equation (SHE)

$$\partial_t \mathcal{Z}(t,x) = \frac{1}{2}\partial_{xx}\mathcal{Z}(t,x) + \xi(t,x)\mathcal{Z}(t,x).$$

- Well-posed equation. Mild solution

$$\mathcal{Z}(t,x) = \int p(t,x-y)\mathcal{Z}(0,y)dy + \int_0^t \int p(t-s,x-y)\mathcal{Z}(s,y)\xi(s,y)dsdy.$$

- Solution preserves positivity [Mueller 91, Moreno Flores 14].

Hopf-Cole solution

- KPZ equation

$$\partial_t \mathcal{H}(t,x) = \frac{1}{2}\partial_{xx}\mathcal{H}(t,x) + \frac{1}{2}\big(\partial_x \mathcal{H}(t,x)\big)^2 + \xi(t,x)$$

- Take $\mathcal{Z} = e^{\mathcal{H}}$, formally $\mathcal{Z}$ solves the SHE

$$\partial_t \mathcal{Z}(t,x) = \frac{1}{2}\partial_{xx}\mathcal{Z}(t,x) + \mathcal{Z}(t,x)\xi(t,x)$$

  Hopf-Cole solution $\mathcal{H}(t,x) := \log \mathcal{Z}(t,x)$.

- Narrow wedge initial data: $\mathcal{Z}(0,x) = \delta_0(x)$.

One point fluctuation of KPZ

- [Amir-Corwin-Quastel 11]

$$\text{(long time)} \qquad \frac{\mathcal{H}(2t,0)+\frac{t}{12}}{t^{\frac{1}{3}}} \Longrightarrow \text{Tracy Widom GUE} \quad t\to\infty$$

  convergence to the KPZ fixed point under 1:2:3 scaling has recently been established in [Quastel-Sarkar 20], [Virag 20].

- [Amir-Corwin-Quastel 11]

$$\text{(short time)} \qquad \frac{\mathcal{H}(2t,0)+\log(\sqrt{4\pi t})}{t^{\frac{1}{4}}} \Longrightarrow \mathcal{N}(0,\sqrt{\frac{\pi}{2}}), \quad t\to 0$$

- How about Large Deviation Principle (LDP)?

LDP of KPZ, long time regime

- Lower tail LDP [Tsai 18], [Cafasso-Claeys 19]. For $\lambda > 0$,

$$\lim_{t\to\infty} \frac{1}{t^2} \log \mathbb{P}\Big(\mathcal{H}(2t,0) + \frac{t}{12} \leq -\lambda t\Big) = -\Phi(\lambda)$$

- Upper tail LDP [Das-Tsai 19],

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(\mathcal{H}(2t,0) + \frac{t}{12} \geq \lambda t\Big) = -\frac{4}{3}\lambda^{\frac{3}{2}}.$$

- What is the LDP in the short time regime?

Main Result

Theorem (L.-Tsai 20)

There exists $\phi : \mathbb{R} \to \mathbb{R}$, *for any* $\lambda > 0$,

$$\lim_{t \to 0} \sqrt{t} \log \mathbb{P}\big(\mathcal{H}(2t, 0) + \log(\sqrt{4\pi t}) > \lambda\big) = -\phi(\lambda),$$
$$\lim_{t \to 0} \sqrt{t} \log \mathbb{P}\big(\mathcal{H}(2t, 0) + \log(\sqrt{4\pi t}) < -\lambda\big) = -\phi(-\lambda),$$

In addition,

$$\lim_{\lambda \to 0} \frac{\phi(\lambda)}{\lambda^2} = \frac{1}{\sqrt{2\pi}}, \qquad \lim_{\lambda \to \infty} \frac{\phi(-\lambda)}{\lambda^{\frac{5}{2}}} = \frac{4}{15\pi},$$

Remark

In [Gaudreau Lamarre-L.-Tsai 21], we showed that

$$\lim_{\lambda \to \infty} \frac{\phi(\lambda)}{\lambda^{\frac{3}{2}}} = \frac{4}{3}.$$

and prove more general result concerning the limit shape of the KPZ equation.

Weak noise scaling

- Consider scaling $\mathcal{H}_\epsilon(t,x) = \mathcal{H}(\epsilon^2 t, \epsilon x) + \log \epsilon$. Then

$$\partial_t \mathcal{H}_\epsilon = \frac{1}{2}\partial_{xx}\mathcal{H}_\epsilon + \frac{1}{2}(\partial_x \mathcal{H}_\epsilon)^2 + \sqrt{\epsilon}\xi$$

- Need to prove

$$\lim_{\epsilon\to 0} \epsilon \log \mathbb{P}\big(\mathcal{H}_\epsilon(2,0) + \log(\sqrt{4\pi}) \geq \lambda\big) = -\phi(\lambda)$$
$$\lim_{\epsilon\to 0} \epsilon \log \mathbb{P}\big(\mathcal{H}_\epsilon(2,0) + \log(\sqrt{4\pi}) \leq -\lambda\big) = -\phi(-\lambda)$$

This is equivalent to proving the LDP of $\mathcal{H}(2t,0) + \log(\sqrt{4\pi t})$ as $t \to 0$.

Rate function

Define $\mathcal{S}(\rho) = \log \mathbb{E}_{0\to 0}\Big[\exp\big(\int_0^2 \rho(s, B_{\mathsf{b}}(s))ds\big)\Big]$ for $\rho \in L^2([0,2]\times\mathbb{R})$.

Lemma (L. Tsai 20)

We have for $\lambda \geq 0$,

$$\phi(-\lambda) = \inf\Big\{\frac{1}{2}\|\rho\|_{L^2}^2 : \mathcal{S}(\rho) \leq -\lambda\Big\},$$

$$\phi(\lambda) = \inf\Big\{\frac{1}{2}\|\rho\|_{L^2}^2 : \mathcal{S}(\rho) \geq \lambda\Big\}.$$

Asymptotics of the rate function

- We focus on the deep lower tail, showing $\phi(-\lambda) \sim \frac{4}{15\pi}\lambda^{\frac{5}{2}}$ as $\lambda \to \infty$.

- Scale $\rho \to \lambda\rho(t, \lambda^{-\frac{1}{2}}x)$, then

$$\phi(-\lambda) = \lambda^{\frac{5}{2}} \inf \left\{ \frac{1}{2}\|\rho\|^2 : \mathcal{S}_\lambda(\rho) \leq -1 \right\}$$

  where

$$\mathcal{S}_\lambda(\rho) = \lambda^{-1} \log \mathbb{E}\Big[\exp\big(\lambda \int_0^2 \rho(s, \lambda^{-\frac{1}{2}} B_{\mathsf{b}}(s)) ds \big) \Big]$$

- Only need to show

$$\lim_{\lambda\to\infty} \inf \left\{ \frac{1}{2}\|\rho\|^2 : \mathcal{S}_\lambda(\rho) \leq -1 \right\} = \frac{4}{15\pi}$$

- What does the optimizer converge to as $\lambda \to \infty$?

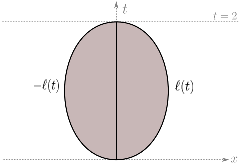
Physics heuristic

- Following [Kamenev-Meerson-Sasorov 16], the limit ρ_* solves the PDEs

$$\partial_t h - \frac{1}{2}(\partial_x h)^2 = \rho$$
$$\partial_t \rho = \partial_x \big(\rho \partial_x h\big)$$

- More precisely

$$\rho_*(t,x) = -\frac{1}{2\pi\ell(t)}\big(1 - \frac{x^2}{\ell(t)^2}\big)\mathbf{1}_{\{|x|\le \ell(t)\}}, \quad \frac{1}{2}\|\rho_*\|_{L^2}^2 = \frac{4}{15\pi}.$$



support of ρ_*

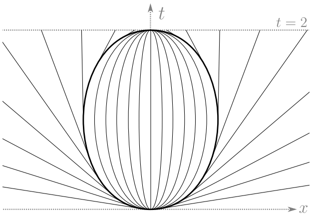
Varadhan's lemma

- By Varadhan's lemma

$$\lim_{\lambda\to\infty} \mathcal{S}_\lambda(\rho_*) = \lim_{\lambda\to\infty} \lambda^{-1} \log \mathbb{E}_{0\to 0}\Big[\exp\big(\lambda \int_0^2 \rho_*(s, \lambda^{-\frac{1}{2}} B_{\mathsf{b}}(s))ds\big)\Big]$$
$$= \sup_{x(0)=x(2)=0} \Big(\int_0^2 \rho_*(s, x(s)) - \frac{1}{2}\dot{x}(s)^2 ds \Big)$$

- Euler-Lagrange equation

$$\ddot{x}(s) + \partial_x \rho_*(s, x(s)) = 0$$



Geodesic (solution) is not unique.

$x(t) = \alpha \ell(t), \alpha \in [-1, 1]$.

They recover ρ_*.

Thank you