Lyapunov exponent of the SHE with general initial data

Yier Lin

Columbia University

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joint work with Promit Ghosal (MIT)

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Stochastic heat equation

- $1+1$ dimension Stochastic heat equation (SHE)

$$\partial_t \mathcal{Z}(t,x) = \frac{1}{2}\partial_{xx}\mathcal{Z}(t,x) + \mathcal{Z}(t,x)\xi(t,x)$$

  $\xi(t,x)$ is space-time white noise.

- Mild Solution

$$\mathcal{Z}(t,x) = \int p(t,x-y)\mathcal{Z}(0,y)dy + \int_0^t \int p(t-s,x-y)\mathcal{Z}(s,y)\xi(s,y)dsdy$$

  For suitable initial $\mathcal{Z}(0,\cdot)$, the solution exists and is unique.

- Strict positivity [Mueller 91], [Moreno Flores 14]

An informal Feynman-Kac

- $1+1$ dimension Stochastic heat equation (SHE)

$$\partial_t \mathcal{Z}(t,x) = \frac{1}{2}\partial_{xx}\mathcal{Z}(t,x) + \mathcal{Z}(t,x)\xi(t,x), \quad \mathcal{Z}(0,\cdot) = f$$

- Feynman Kac

$$\mathcal{Z}(t,x) = \mathbb{E}_x\Big[\exp\Big(\int_0^t \xi(t-s, B(s))\Big) f(B(t))\Big]$$

- When f is a Dirac-Delta function,

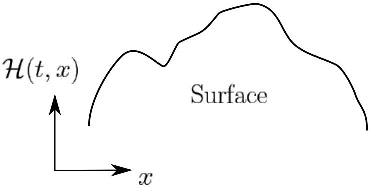
$$\mathcal{Z}(t,x) = \mathbb{E}_{x\to 0}\Big[\exp\Big(\int_0^t \xi(t-s, B_b(s))\Big)\Big] p(t,x)$$

Consequently, $\mathcal{Z}(t,x)e^{\frac{x^2}{2t}}$ is stationary in x.

KPZ equation

- Kardar-Parisi-Zhang equation

$$\partial_t \mathcal{H}(t,x) = \frac{1}{2}\partial_{xx}\mathcal{H}(t,x) + \frac{1}{2}(\partial_x \mathcal{H}(t,x))^2 + \xi(t,x)$$



- Hopf-Cole solution $\mathcal{H}(t,x) = \log \mathcal{Z}(t,x)$

$$\partial_t \mathcal{Z}(t,x) = \frac{1}{2}\partial_{xx}\mathcal{Z}(t,x) + \mathcal{Z}(t,x)\xi(t,x)$$

Initial data

- Two types of initial data
  - $\mathcal{Z}(0,x) = \delta(x)$ narrow wedge
  - $\mathcal{Z}(0,x) = \exp(f(x))$, $f : \mathbb{R} \to \mathbb{R}$ general initial data

- As a notation, we will denote the SHE with above initial data to be $\mathcal{Z}^{\mathbf{nw}}$ and $\mathcal{Z}^{f}$, respectively. For the corresponding KPZ equation, denote by $\mathcal{H}^{\mathbf{nw}}$ and $\mathcal{H}^{f}$.

Lyapunov exponent

- For $p > 0$, define the p-th Lyapunov exponent to be

$$\gamma(p) := \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}\Big[\mathcal{Z}(t,0)^p\Big]$$

  if the limit exists.

- The Lyapunov exponent is related to
  - Intermittency: $\gamma(1) < \frac{\gamma(2)}{2} < \frac{\gamma(3)}{3} < \dots$
  - Large deviation principle (LDP)

Intermittency

- Intermittency: $\gamma(1) < \frac{\gamma(2)}{2} < \frac{\gamma(3)}{3} < \dots$
- Set $\frac{\gamma(n-1)}{n-1} < \beta_n < \frac{\gamma(n)}{n}$. Consider the random set,

$$\mathcal{A}_n = \{\mathcal{Z}(t,0) \geq e^{t\beta_n}\}$$

Clearly, we have

$$\mathcal{A}_1 \supseteq \mathcal{A}_2 \supseteq \mathcal{A}_3 \supseteq \dots$$

Moreover, $\mathbb{P}(\mathcal{A}_n)$ decays exponentially in t. However, $\mathcal{A}_n$ explains the n-th moment of $\mathcal{Z}(t,0)$

$$\underbrace{\mathbb{E}\Big[\mathcal{Z}(t,0)^n\Big]}_{\sim \exp(t\gamma(n))} = \mathbb{E}\Big[\mathcal{Z}(t,0)^n \mathbb{1}_{\mathcal{A}_n}\Big] + \underbrace{\mathbb{E}\Big[\mathcal{Z}(t,0)^n \mathbb{1}_{\mathcal{A}_n^c}\Big]}_{\leq \exp(tn\beta_n)}$$

LDP

- [Amir-Corwin-Quastel 11] As $t \to \infty$,

$$(t/2)^{-\frac{1}{3}}(\mathcal{H}^{\mathbf{nw}}(t,0) + \frac{t}{24}) \Rightarrow \text{Tracy-Widom GUE}$$

[Das-Tsai 19] For $p > 0$,

$$\gamma^{\mathbf{nw}}(p) = \frac{1}{t}\log \mathbb{E}\Big[\mathcal{Z}^{\mathbf{nw}}(t,0)^p\Big] = \frac{p^3}{24} - \frac{p}{24}$$

They also get the upper tail LDP,

$$\lim_{t\to\infty} \frac{1}{t}\log \mathbb{P}\Big(\mathcal{H}^{\mathbf{nw}}(t,0) + \frac{t}{24} > st\Big) = -\frac{4\sqrt{2}}{3}s^{\frac{3}{2}}$$

Note that $\frac{4\sqrt{2}}{3}s^{\frac{3}{2}} = \max_{p>0}\left(sp - \frac{p^3}{24}\right)$

LDP

Lemma (Ghosal-L. 20)

Let $X(t)$ be a stochastic process. Assume there exists $h : C^1(\mathbb{R}_{>0})$ such that $h' : (0, \infty) \to (\zeta, \infty)$ is bijective and increasing and for every $p > 0$,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[e^{pX(t)}\right] = h(p)$$

Then for any $s > \zeta$

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{P}\Big(X(t) > st\Big) = -\sup_{p>0} \Big\{ps - h(p)\Big\}$$

Lyapunov exponent

- We review some technique in the computation of the Lyapunov exponent.
- [Bertini-Cancrini 95], [Hu-Nualart 11]

$$\mathbb{E}\Big[\mathcal{Z}(t,0)^n\Big] = \mathbb{E}\Big[\exp\Big(\sum_{1\leq i<j\leq n}\int_0^t \delta(B_i - B_j)\Big)\Big]$$

claim that $\gamma(n) = \frac{n^3-n}{24}$ for $n \in \mathbb{Z}_{\geq 1}$.

- Analyzing the above formula, [Chen 15] provides a rigorous proof of $\gamma(n) = \frac{n^3-n}{24}$ for $n \in \mathbb{Z}_{\geq 1}$.
- [Corwin-Ghosal 18] shows that for $\mathcal{Z}(0,\cdot) = \delta(\cdot)$, $\gamma^{\textbf{nw}}(n) = \frac{n^3-n}{24}$

$$\mathbb{E}\Big[\mathcal{Z}^{\textbf{nw}}(t,0)^n\Big] = n \text{ fold integral}$$

Lyapunov exponent

[Das-Tsai 19] shows that $\gamma^{\textsf{nw}}(p) = \frac{p^3 - p}{24}$ for $p \in \mathbb{R}_{>0}$. Consequently, they obtain

$$\frac{1}{t} \log \mathbb{P}\Big(\mathcal{H}^{\textsf{nw}}(t,0) + \frac{t}{24} > st\Big) = -\frac{4\sqrt{2}}{3} s^{\frac{3}{2}}.$$

Proof sketch: By integrability [Amir-Corwin-Quastel 11], we have

$$\mathbb{E}\Big[\exp\Big(-s\mathcal{Z}^{\textsf{nw}}(t,0)e^{\frac{t}{24}}\Big)\Big] = \det\Big[I - K_{s,t}\Big], s > 0 \qquad (1)$$

Lemma

For $\alpha \in [0,1)$, we have

$$\mathbb{E}\Big[U^{n-1+\alpha}\Big] = \frac{(-1)^n}{\Gamma(1-\alpha)} \int_0^\infty s^{-\alpha} \frac{d^n}{ds^n} \mathbb{E}\Big[e^{-sU}\Big] ds$$

For the SHE $\mathcal{Z}^f$ under initial data $\mathcal{Z}^{\textsf{nw}}(0,\cdot) = \exp(f(\cdot))$, there is no exact formula like (1).

Lyapunov exponent

Denote $\gamma^f(p) := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[\mathcal{Z}^f(t,0)^p\right]$. Our result applies to a general class of initial data. For simplicity, let me present two special cases.

Theorem (Ghosal L. 20)

If f is bounded, then $\gamma^f(p) = \frac{p^3-p}{24}$ for all $p > 0$. Moreover,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\left(\mathcal{H}^f(t,0) + \frac{t}{24} > st\right) = -\frac{4\sqrt{2}}{3} s^{\frac{3}{2}}.$$

Theorem (Ghosal L. 20)

If f is a two sided Brownian motion, then $\gamma^f(p) = \frac{4p^3-p}{24}$ for all $p > 0$. Moreover, for $s > 0$,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\left(\mathcal{H}^f(t,0) + \frac{t}{24} > st\right) = -\frac{2\sqrt{2}}{3} s^{\frac{3}{2}}.$$

Proof Ingredients

- [Corwin-Hammond 16] distributional identity

$$\mathcal{Z}^f(t,0) = \int \mathcal{Z}^{\textbf{nw}}(t,x)\exp(f(x))dx$$

  where f and $\mathcal{Z}^{\textbf{nw}}$ are independent.

- [Amir-Corwin-Quastel 11] $\mathcal{Z}^{\textbf{nw}}(t,x)\exp(\frac{x^2}{2t})$ is stationary in x.

- Spatial increment tail bound [Corwin-Ghosal-Hammond 19]

Heuristic

- We assume the existence of $g : \mathbb{R}_{>0} \to \mathbb{R}$ such that for every $p > 0$,

$$g(p) := \lim_{t \to \infty} \frac{1}{t} \log \int \mathbb{E}\Big[e^{pf(x)} e^{-\frac{px^2}{2t}} dx\Big]$$

- Since

$$\mathcal{Z}^f(t,0) \overset{d}{=} \int \mathcal{Z}^{\mathbf{nw}}(t,x) e^{f(x)} dx,$$

we have

$$\mathbb{E}\big[\mathcal{Z}^f(t,0)^p\big] = \mathbb{E}\Big[\Big(\int \mathcal{Z}^{\mathbf{nw}}(t,x) e^{f(x)} dx\Big)^p\Big]$$

Heuristic

- Approximation

$$\begin{aligned}\mathbb{E}\big[\mathcal{Z}^f(t,0)^p\big] &= \mathbb{E}\Big[\Big(\int \mathcal{Z}^{\textbf{nw}}(t,x)e^{f(x)}dx\Big)^p\Big] \\ &\approx \mathbb{E}\Big[\int \mathcal{Z}^{\textbf{nw}}(t,x)^p e^{pf(x)}dx\Big] \\ &= \int \mathbb{E}\big[\mathcal{Z}^{\textbf{nw}}(t,x)^p\big]\mathbb{E}\big[e^{pf(x)}\big]dx \\ &= \mathbb{E}\big[\mathcal{Z}^{\textbf{nw}}(t,0)^p\big]\int \mathbb{E}\big[e^{p(f(x)-\frac{x^2}{2t})}\big]dx\end{aligned}$$

- The last step follows from the stationarity of $\mathcal{Z}^{\textbf{nw}}(t,x)e^{\frac{x^2}{2t}}$.
- We conclude $\gamma^f(p) = \gamma^{\textbf{nw}}(p) + g(p)$
- f is bounded, $g = 0$; f is Brownian, $g(p) = \frac{p^3}{8}$.

Technical condition of the initial data

'Condition' in the heuristic

$\lim_{t\to\infty} \frac{1}{t} \log \left(\int \mathbb{E}\left[e^{pf(x)}\right] e^{-\frac{px^2}{2t}} dx \right) = g(p)$.

Real condition

Consider f satisfying the conditions: there exists $g : \mathbb{R}_{>0} \to \mathbb{R}$ s.t. for all $p > 0$,

1. $$\lim_{t\to\infty} \frac{1}{t} \sup_{x\in\mathbb{R}} \left(\log \mathbb{E}\left[e^{pf(x)}\right] - \frac{px^2}{2t} \right) = g(p)$$
2. $$\liminf_{\epsilon\to 0} \lim_{t\to\infty} \frac{1}{t} \log \int e^{-\frac{p(1-\epsilon)x^2}{2t}} \mathbb{E}\left[e^{p(1+\epsilon)f(x)}\right] dx \le g(p)$$
3. There exists $C, \kappa, \delta > 0$ s.t. for all $s > 0$ and x, y s.t. $|x - y| \le 1$
$$\mathbb{P}\Big(|f(x) - f(y)| > s\Big) \le C \exp(-\kappa s^{1+\delta})$$

Result

Theorem (Ghosal, L.20)

Assume f satisfies the previous condition, then for every $p > 0$, $\gamma_p^f = \frac{p^3 - p}{24} + g(p)$. Furthermore, if $g \in C^1(\mathbb{R}_{>0})$ and $\lim_{p \to 0} g'(p) = \zeta$ is finite, then for $s > \zeta$,

$$\lim_{t \to \infty} \frac{1}{t} \mathbb{P}\Big(\mathcal{H}^f(t, 0) + \frac{t}{24} > st\Big) = -I(s)$$

where $I(s) = \max_{p \geq 0}\{sp - \frac{p^3}{24} - g(p)\}$

Proof

Goal

(lower bound) $\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}\big[\mathcal{Z}^f(t,0)^p\big] \geq g(p) + \gamma^{\mathbf{nw}}(p)$.

(upper bound) $\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}\big[\mathcal{Z}^f(t,0)^p\big] \leq g(p) + \gamma^{\mathbf{nw}}(p)$

Proof (Upper bound $p > 1$)

By Hölder inequality, $q = \frac{p}{p-1}$,

$$\mathcal{Z}^f(t,0) = \int e^{-\frac{\epsilon x^2}{2t}} \Big(e^{\frac{\epsilon x^2}{2t}} \mathcal{Z}^{\mathbf{nw}}(t,x) e^{f(x)}\Big) dx$$
$$\leq \Big(\int e^{-\frac{q\epsilon x^2}{2t}} dx\Big)^{\frac{1}{q}} \Big(\int \mathcal{Z}^{\mathbf{nw}}(t,x)^p e^{\frac{p\epsilon x^2}{2t}} e^{pf(x)} dx\Big)^{\frac{1}{p}}$$

Upper bound $p > 1$

Proof (continued)

$$\mathcal{Z}^f(t,0) \le \Big(\int e^{-\frac{q\epsilon x^2}{2t}}dx\Big)^{\frac{1}{q}}\Big(\int \mathcal{Z}^{\mathsf{nw}}(t,x)^p e^{\frac{p\epsilon x^2}{2t}} e^{pf(x)}dx\Big)^{\frac{1}{p}}$$

$$\Longrightarrow \mathbb{E}\Big[\mathcal{Z}^f(t,0)^p\Big] \le \Big(\frac{2\pi t}{q\epsilon}\Big)^{\frac{p}{2q}}\mathbb{E}\Big[\int \mathcal{Z}^{\mathsf{nw}}(t,x)^p e^{\frac{p\epsilon x^2}{2t}} e^{pf(x)}dx\Big]$$

Recall $\mathbb{E}\big[\mathcal{Z}^{\mathsf{nw}}(t,x)^p\big] = \mathbb{E}\big[\mathcal{Z}^{\mathsf{nw}}(t,0)^p\big]e^{-\frac{px^2}{2t}}$,

$$\mathbb{E}\big[\mathcal{Z}^f(t,0)^p\big] \le \Big(\frac{2\pi t}{q\epsilon}\Big)^{\frac{p}{2q}}\mathbb{E}\big[\mathcal{Z}^{\mathsf{nw}}(t,0)^p\big]\int e^{-\frac{p(1-\epsilon)x^2}{2t}}\mathbb{E}\big[e^{pf(x)}\big]dx$$

By condition, we conclude

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\big[\mathcal{Z}^f(t,0)^p\big] \le \gamma^{\mathsf{nw}}(p) + g(p).$$

Upper bound, $p \in (0, 1]$

Due to sub-additivity, for $p \leq 1$ and a_n positive,

$$\Big(\sum_n a_n\Big)^p \leq \sum_n a_n^p$$

Then

$$\begin{aligned}\mathbb{E}\Big[\mathcal{Z}^f(t,0)^p\Big] &= \mathbb{E}\bigg[\Big(\int \mathcal{Z}^{\textsf{nw}}(t,x)e^{f(x)}dx\Big)^p\bigg] \\ &\leq \sum_{n\in\mathbb{Z}} \mathbb{E}\Big[\Big(\int_n^{n+1} \mathcal{Z}^{\textsf{nw}}(t,x)e^{f(x)}dx\Big)^p\Big].\end{aligned}$$

Need to compare $\mathcal{Z}^{\textsf{nw}}(t,x)$ with $\mathcal{Z}^{\textsf{nw}}(t,y)$ when x, y satisfy $|x-y| \leq 1$.

Increment tail bound

Proposition (Corwin-Ghosal-Hammond 19)

Recall $\mathcal{H}^{\textsf{nw}} = \log \mathcal{Z}^{\textsf{nw}}$. *For any* $\epsilon > 0$, *there exists* C, κ *s.t. for all* $t > 1$ *and* $s > 0$

$$\mathbb{P}\Big(\sup_{x \in [0, t^{\frac{1}{3}}]} \big(\mathcal{H}^{\textsf{nw}}(t,x) - \mathcal{H}^{\textsf{nw}}(t,0) - \frac{x^2}{2}\big) \geq s \Big) \leq C e^{-\kappa s^{\frac{9}{8} - \epsilon}}$$

$$\mathbb{P}\Big(\inf_{x \in [0, t^{\frac{1}{3}}]} \big(\mathcal{H}^{\textsf{nw}}(t,x) - \mathcal{H}^{\textsf{nw}}(t,0) + \frac{x^2}{2}\big) \leq -s \Big) \leq C e^{-\kappa s^{\frac{9}{8} - \epsilon}}$$

Lemma

Assume $\mathbb{P}(Y_t > s) \leq C \exp(-\kappa s^{1+\delta})$ *for all* $t, s > 1$. *Let* X *be a non-negative random variable. For any fixed* $\epsilon > 0$, $\exists C$ *s.t. for all* t,

$$\mathbb{E}[X e^{Y_t}] \leq C \big(\mathbb{E}[X^{1+\epsilon}]\big)^{\frac{1}{1+\epsilon}}$$

Upper bound, $p \in (0,1]$

$$\mathbb{E}\big[\mathcal{Z}^f(t,0)^p\big] \le \textstyle\sum_{n\in\mathbb{Z}} \underbrace{\mathbb{E}\Big[\Big(\displaystyle\int_n^{n+1} \mathcal{Z}^{\mathbf{nw}}(t,x)e^{f(x)}dx\Big)^p\Big]}_{\mathcal{I}_n}.$$

$$\mathcal{J}_n(t) := \sup\nolimits_{x\in[n,n+1]} \big(\mathcal{H}^{\mathbf{nw}}(t,x) - \mathcal{H}^{\mathbf{nw}}(t,n) + \tfrac{(x-n)n}{t} - \tfrac{(x-n)^2}{2}\big)$$

$\big(\mathcal{Z}^{\mathbf{nw}}(t,n)e^{\frac{n^2}{2t}}, \mathcal{J}_n(t)\big)$ is stationary in n. Recall that $\mathcal{Z}^{\mathbf{nw}} = \exp(\mathcal{H}^{\mathbf{nw}})$,

$$\mathcal{Z}^{\mathbf{nw}}(t,x) \le C\mathcal{Z}^{\mathbf{nw}}(t,n)e^{\mathcal{J}_n(t)}, \qquad x \in [n,n+1]$$

Hence,

$$\begin{aligned}\mathcal{I}_n &\le C\mathbb{E}\big[\mathcal{Z}^{\mathbf{nw}}(t,n)^p e^{p\mathcal{J}_n(t)}\big]\mathbb{E}\Big[\Big(\int_n^{n+1} e^{f(x)}dx\Big)^p\Big] \\ &= Ce^{-\frac{pn^2}{2t}}\mathbb{E}\big[\mathcal{Z}^{\mathbf{nw}}(t,0)^p e^{p\mathcal{J}_0(t)}\big])\mathbb{E}\Big[\Big(\int_n^{n+1} e^{f(x)}dx\Big)^p\Big]\end{aligned}$$

Upper bound, $p \in (0,1]$

- $\mathbb{E}\Big[\mathcal{Z}^f(t,0)^p\Big] \leq \sum_{n\in\mathbb{Z}} \underbrace{\mathbb{E}\Big[\Big(\int_n^{n+1} \mathcal{Z}^{\mathbf{nw}}(t,x)e^{f(x)}dx\Big)^p\Big]}_{\mathcal{I}_n}.$

- $\mathcal{I}_n \leq Ce^{-\frac{pn^2}{2t}}\mathbb{E}\big[\mathcal{Z}^{\mathbf{nw}}(t,0)^p e^{p\mathcal{J}_0(t)}\big])\mathbb{E}\Big[\Big(\int_n^{n+1} e^{f(x)}dx\Big)^p\Big]$

- This implies that

$$\mathbb{E}\big[\mathcal{Z}^f(t,0)^p\big] \leq C\underbrace{\mathbb{E}\big[\mathcal{Z}^{\mathbf{nw}}(t,0)^p e^{p\mathcal{J}_0(t)}\big]}_{A_1(t)} \times \underbrace{\Big(\sum_n e^{-\frac{pn^2}{2t}}\mathbb{E}\Big[\Big(\int_n^{n+1} e^{f(x)}dx\Big)^p\Big]\Big)}_{A_2(t)}$$

Upper bound, $p \in (0,1]$

$$\mathbb{E}\big[\mathcal{Z}^f(t,0)^p\big] \le C \underbrace{\mathbb{E}\big[\mathcal{Z}^{\mathsf{nw}}(t,0)^p e^{p\mathcal{J}_0(t)}\big]}_{A_1(t)} \times$$

$$\underbrace{\Big(\sum_n e^{-\frac{pn^2}{2t}} \mathbb{E}\Big[\Big(\int_n^{n+1} e^{f(x)}dx\Big)^p\Big]\Big)}_{A_2(t)}$$

$$A_1(t) \le C\Big(\mathbb{E}\big[\mathcal{Z}^{\mathsf{nw}}(t,0)^{p(1+\epsilon)}\big]\Big)^{\frac{1}{1+\epsilon}}$$

$$A_2(t) \le C \int e^{-\frac{p(1-\epsilon)x^2}{2t}} \mathbb{E}\big[e^{p(1+\epsilon)f(x)}\big]dx$$

Consequently,

$$\limsup \frac{1}{t}\log A_1(t) \le \frac{p^3-p}{24} \qquad \text{[Das-Tsai 19]}$$

$$\limsup \frac{1}{t}\log A_2(t) \le g(p) \qquad \text{our condition}$$

Half-line SHE

Consider the half-space SHE with Robin boundary

$$\partial_t \mathcal{Z}(t,x) = \frac{1}{2}\partial_{xx}\mathcal{Z}(t,x) + \xi(t,x)\mathcal{Z}(t,x)$$
$$\partial_x \mathcal{Z}(t,0) = A\mathcal{Z}(t,0), \qquad \mathcal{Z}(0,x) = \delta(x).$$

When $A = -\frac{1}{2}$, [Barraquand-Borodin-Corwin-Wheeler 18], [Parekh 19] shows that

$$(t/2)^{-\frac{1}{3}}\left(\log \mathcal{Z}(t,0) + \frac{t}{24}\right) \Rightarrow \text{Tracy-Widom GOE}$$

Theorem (L. 20)

For the half-line SHE with $A = -\frac{1}{2}$, for every $p > 0$, $\gamma(p) = \frac{4p^3 - p}{24}$. Furthermore,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\left(\mathcal{H}(t,0) + \frac{t}{24} > st\right) = -\frac{2\sqrt{2}}{3} s^{\frac{3}{2}}.$$

Thank you!