Short time large deviations of the KPZ equation

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Mar 22nd, 2022

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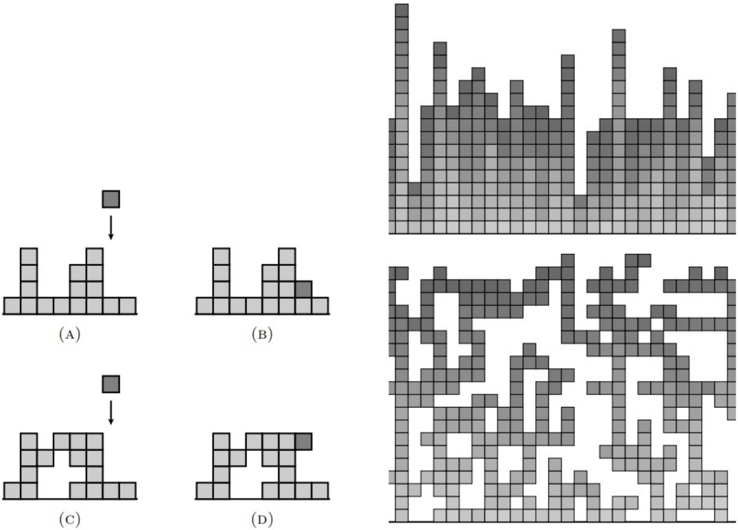
Ballistic decomposition and random growth model



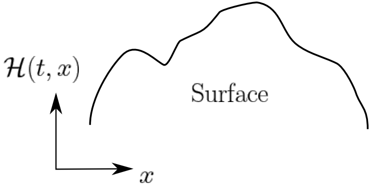
(Picture Courtesy of Ivan Corwin)

KPZ equation

- Introduced by Kardar-Parisi-Zhang (1986)

$$\partial_t \mathcal{H}(t,x) = \frac{1}{2}\partial_{xx}\mathcal{H}(t,x) + \frac{1}{2}\big(\partial_x \mathcal{H}(t,x)\big)^2 + \xi(t,x)$$

ξ is the space-time white noise (Gaussian field with correlation function $E[\xi(t,x)\xi(s,y)] = \delta(t-s)\delta(x-y)$).



Stochastic heat equation

- Stochastic heat equation (SHE)

$$\partial_t \mathcal{Z}(t,x) = \frac{1}{2}\partial_{xx}\mathcal{Z}(t,x) + \xi(t,x)\mathcal{Z}(t,x).$$

- Well-posed equation. Mild solution

$$\mathcal{Z}(t,x) = \int p(t,x-y)\mathcal{Z}(0,y)dy + \int_0^t \int p(t-s,x-y)\mathcal{Z}(s,y)\xi(s,y)dsdy.$$

- Solution preserves positivity.

Hopf-Cole solution

- KPZ equation

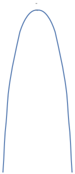
$$\partial_t \mathcal{H}(t,x) = \frac{1}{2}\partial_{xx}\mathcal{H}(t,x) + \frac{1}{2}\big(\partial_x \mathcal{H}(t,x)\big)^2 + \xi(t,x)$$

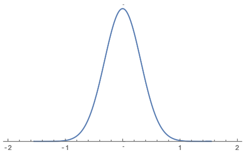
- Take $\mathcal{Z} = e^{\mathcal{H}}$, formally $\mathcal{Z}$ solves the SHE

$$\partial_t \mathcal{Z}(t,x) = \frac{1}{2}\partial_{xx}\mathcal{Z}(t,x) + \mathcal{Z}(t,x)\xi(t,x)$$

Hopf-Cole solution $\mathcal{H}(t,x) := \log \mathcal{Z}(t,x)$.

- Narrow wedge initial data: $\mathcal{Z}(0,x) = \delta_0(x)$, $\mathcal{Z}(t,x) \approx p(t,x)$.



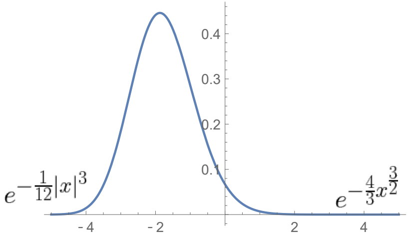


One point fluctuation of KPZ

- [Amir-Corwin-Quastel 11]

(long time) $$\frac{\mathcal{H}(2t,0)+\frac{t}{12}}{t^{\frac{1}{3}}} \Longrightarrow \text{Tracy Widom GUE} \quad t\to\infty$$

(short time) $$\frac{\mathcal{H}(2t,0)+\log(\sqrt{4\pi t})}{t^{\frac{1}{4}}} \Longrightarrow \mathcal{N}(0,\sqrt{\frac{\pi}{2}}), \quad t\to 0$$



pdf of Tracy Widom-GUE

- What about Large Deviation Principle (LDP)?

LDP of KPZ, long time regime

- Lower tail LDP [Tsai 18], [Cafasso-Claeys 19]. For $\lambda > 0$,

$$\lim_{t\to\infty} \frac{1}{t^2} \log \mathbb{P}\Big(\mathcal{H}(2t,0) + \frac{t}{12} \leq -\lambda t\Big) = -\Phi_-(\lambda)$$

$\Phi_-(\lambda)$ explicit, $\lim_{\lambda\to 0^+} \frac{\Phi_-(\lambda)}{\lambda^3} = \frac{1}{12}$, $\lim_{\lambda\to\infty} \frac{\Phi_-(\lambda)}{\lambda^{\frac{5}{2}}} = \frac{4}{15\pi}$

  - For finite large t, a tail bound which identifies this crossover was established earlier in [Corwin-Ghosal 18]

- Upper tail LDP [Das-Tsai 19],

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(\mathcal{H}(2t,0) + \frac{t}{12} \geq \lambda t\Big) = -\frac{4}{3}\lambda^{\frac{3}{2}}.$$

- What is the LDP in the short time regime?

Main Result

Theorem (L.-Tsai 20, Gaudreau Lamarre-L.-Tsai 21)

There exists $\phi : \mathbb{R} \to \mathbb{R}$, *for any* $\lambda > 0$,

$$\lim_{t \to 0} \sqrt{t} \log \mathbb{P}\big(\mathcal{H}(2t, 0) + \log(\sqrt{4\pi t}) > \lambda\big) = -\phi(\lambda),$$

$$\lim_{t \to 0} \sqrt{t} \log \mathbb{P}\big(\mathcal{H}(2t, 0) + \log(\sqrt{4\pi t}) < -\lambda\big) = -\phi(-\lambda),$$

In addition,

$$\lim_{\lambda \to 0} \frac{\phi(\lambda)}{\lambda^2} = \frac{1}{\sqrt{2\pi}}, \qquad \lim_{\lambda \to \infty} \frac{\phi(-\lambda)}{\lambda^{\frac{5}{2}}} = \frac{4}{15\pi}, \qquad \lim_{\lambda \to \infty} \frac{\phi(\lambda)}{\lambda^{\frac{3}{2}}} = \frac{4}{3}$$

In [Gaudreau Lamarre-L.-Tsai 21], we also prove more general result concerning the limit shape of the KPZ equation, this is not the focus for today.

Weak noise scaling

- Consider scaling $\mathcal{H}_\epsilon(t,x) = \mathcal{H}(\epsilon^2 t, \epsilon x) + \log\epsilon$. Then

$$\partial_t \mathcal{H}_\epsilon = \frac{1}{2}\partial_{xx}\mathcal{H}_\epsilon + \frac{1}{2}(\partial_x \mathcal{H}_\epsilon)^2 + \sqrt{\epsilon}\xi$$

- Need to prove

$$\lim_{\epsilon\to 0} \epsilon \log \mathbb{P}\big(\mathcal{H}_\epsilon(2,0) + \log(\sqrt{4\pi}) \geq \lambda\big) = -\phi(\lambda)$$
$$\lim_{\epsilon\to 0} \epsilon \log \mathbb{P}\big(\mathcal{H}(2,0) + \log(\sqrt{4\pi}) \leq -\lambda\big) = -\phi(-\lambda)$$

  This is equivalent to proving the LDP of $\mathcal{H}(2t,0) + \log(\sqrt{4\pi t})$ as $t \to 0$.
- Study the functional LDP of $\mathcal{H}_\epsilon(\cdot,\cdot)$ on $[0,2]\times\mathbb{R}$ as $\epsilon \to 0$.

Functional LDP of SHE

- Consider $\mathcal{Z}_\epsilon = e^{\mathcal{H}_\epsilon}$, then

$$\partial_t \mathcal{Z}_\epsilon = \frac{1}{2}\partial_{xx}\mathcal{Z}_\epsilon + \sqrt{\epsilon}\xi\mathcal{Z}_\epsilon, \quad \mathcal{Z}_\epsilon(0,\cdot) = \delta_0(\cdot)$$

- Feynman-Kac formula

$$\frac{\mathcal{Z}_\epsilon(t,x)}{p(t,x)} = \mathbb{E}_{0\to x}\Big[\exp\Big(\int_0^t \sqrt{\epsilon}\xi(s, B_{\mathsf{b}}(s))ds\Big)\Big]$$

- Formal derivation of functional LDP of $\frac{Z_\epsilon(\cdot,\cdot)}{p(\cdot,\cdot)} \in C([0,2]\times\mathbb{R})$: Consider

$$\mathcal{J}(\rho) = \mathbb{E}_{0\to x}\Big[\exp\Big(\int_0^t \rho(s, B_{\mathsf{b}}(s))ds\Big)\Big]$$

One should expect $\mathcal{J}(\sqrt{\epsilon}\xi) = \frac{\mathcal{Z}_\epsilon(\cdot,\cdot)}{p(\cdot,\cdot)}$ satisfies a LDP with speed ϵ^{-1} and rate function

$$I(f) = \inf\Big\{\frac{1}{2}\|\rho\|^2 : \rho \in L^2([0,2]\times\mathbb{R}), \mathcal{J}(\rho) = f\Big\}$$

- Rigorous proof. Chao series expansion

$$\frac{\mathcal{Z}_\epsilon(t,x)}{p(t,x)} = \sum_{k=0}^{\infty} \mathcal{I}_k(\sqrt{\epsilon}\xi)(t,x)$$

where

$$\mathcal{I}_k(\sqrt{\epsilon}\xi)(t,x) = \int P_k(t,x,t_1,x_1,\ldots,t_k,x_k) \prod_{i=1}^{k} \sqrt{\epsilon}\xi(t_i,x_i)$$

Functional LDP of SHE

- [Hairer-Weber 15] LDP of $\sum_{k=0}^{N} \mathcal{I}_k(\sqrt{\epsilon}\xi)(\cdot,\cdot)$ with speed ϵ^{-1} has rate

$$I(f) = \inf\left\{\frac{1}{2}\|\rho\|^2 : \rho \in L^2([0,2]\times\mathbb{R}), \sum_{k=0}^{N} \mathcal{I}_k(\rho) = f\right\}$$

- Exponential approximation
  - Hypercontractive inequality $\Rightarrow$ One point tail estimate of $\mathcal{I}_k(\sqrt{\epsilon}\xi)(t,x)$
  - Kolmogorov type argument $\Rightarrow$ Functional tail estimate of $\mathcal{I}_k(\sqrt{\epsilon}\xi)(\cdot,\cdot)$
- Functional LDP of $\frac{\mathcal{Z}_\epsilon(\cdot,\cdot)}{p(\cdot,\cdot)} = \sum_{k=0}^{\infty} \mathcal{I}_k(\sqrt{\epsilon}\xi)(\cdot,\cdot)$

$$I(f) = \inf\left\{\frac{1}{2}\|\rho\|^2 : \rho \in L^2([0,2]\times\mathbb{R}), \sum_{k=0}^{\infty} \mathcal{I}_k(\rho) = f\right\}$$

- Feynman-Kac formula

$$\sum_{k=0}^{\infty} \mathcal{I}_k(\rho)(t,x) = \mathbb{E}_{0\to x}\left[\exp\Big(\int_0^t \rho(s, B_{\mathsf{b}}(s))ds\Big)\right]$$

Rate function

Theorem (L. Tsai 20)

$\mathcal{Z}_\epsilon(\cdot,\cdot)/p(\cdot,\cdot)$ *satisfies a LDP with speed* ϵ^{-1} *and a good rate function*

$$I(f) = \inf\left\{\frac{1}{2}\|\rho\|^2 : \mathcal{J}(\rho) = f, \rho \in L^2([0,2]\times\mathbb{R})\right\}$$

$$\mathcal{J}(\rho)(t,x) = \mathbb{E}_{0\to x}\Big[\exp\big(\textstyle\int_0^t \rho(s, B_b(s))ds\big)\Big]$$

We push the functional LDP forward to one point $\mathcal{H}_\epsilon(2,0) + \log\sqrt{4\pi}$ as $\epsilon \to 0$

Corollary (L. Tsai 20)

Let $\mathcal{S}(\rho) = \big(\log \mathcal{J}(\rho)\big)(2,0)$. *We have for* $\lambda \geq 0$,

$$\phi(-\lambda) = \inf\left\{\frac{1}{2}\|\rho\|^2 : \mathcal{S}(\rho) \leq -\lambda\right\}$$

$$\phi(\lambda) = \inf\left\{\frac{1}{2}\|\rho\|^2 : \mathcal{S}(\rho) \geq \lambda\right\}$$

Asymptotics

- We focus on the deep lower tail, showing $\phi(-\lambda) \sim \frac{4}{15\pi}\lambda^{\frac{5}{2}}$ as $\lambda \to \infty$.

  The near centered tail is not hard. The deep upper tail $\phi(\lambda) \sim \frac{4}{3}\lambda^{\frac{3}{2}}$ as $\lambda \to \infty$ follows from a completely different argument.

- LDP of KPZ in short time has been studied in the physics literature [Kolokolov-Korshunov 07], [Kolokolov-Korshunov 09], [Le Doussal-Majumdar-Schehr-Rosso 16], [Kamenev-Meerson-Sasarov 16]

- We explain the heuristic named weak noise theory from [Kamenev-Meerson-Sasarov 16]. This heuristic provides an input for the rigorous proof.

Physics heuristic

- Recall the weak noise KPZ equation

$$\partial_t \mathcal{H}_\epsilon = \frac{1}{2}\partial_{xx}\mathcal{H}_\epsilon + \frac{1}{2}(\partial_x \mathcal{H}_\epsilon)^2 + \sqrt{\epsilon}\xi.$$

- For $\rho \in L^2([0,2] \times \mathbb{R})$,

$$-\epsilon \log \mathbb{P}\big(\sqrt{\epsilon}\xi \approx \rho\big) \approx \frac{1}{2}\|\rho\|^2, \qquad \epsilon \to 0$$

  Note that

$$\partial_t \mathcal{H}_\epsilon - \frac{1}{2}\partial_{xx}\mathcal{H}_\epsilon - \frac{1}{2}(\partial_x \mathcal{H}_\epsilon)^2 = \sqrt{\epsilon}\xi$$

  This implies that

$$-\epsilon \log \mathbb{P}\big(\mathcal{H}_\epsilon \approx h\big) \approx \frac{1}{2}\|\partial_t h - \frac{1}{2}\partial_{xx} h - \frac{1}{2}(\partial_x h)^2\|^2$$

$$\downarrow$$

$$\rho$$

Physics heuristic

- Let $\rho = \partial_t h - \frac{1}{2}\partial_{xx} h - \frac{1}{2}(\partial_x h)^2$, then

$$-\epsilon \log \mathbb{P}\big(\mathcal{H}_\epsilon \approx h\big) \approx \frac{1}{2}\|\rho\|^2, \quad \epsilon \to 0.$$

- We want to find $-\epsilon \log \mathbb{P}\big(\mathcal{H}_\epsilon(2,0) \approx -\lambda\big) \approx -\phi(-\lambda)$ as $\epsilon \to 0$
- Given the boundary $h(2,0) = -\lambda$, minimize

$$F(h) := \frac{1}{2}\int_0^2 \int \big(\partial_t h - \frac{1}{2}\partial_{xx} h - \frac{1}{2}(\partial_x h)^2\big)^2 dt dx.$$

- By calculus of variation, we get

$$\partial_t h - \frac{1}{2}\partial_{xx} h - \frac{1}{2}(\partial_x h)^2 = \rho$$
$$\partial_t \rho + \frac{1}{2}\partial_{xx}\rho = \partial_x\big(\rho \partial_x h\big)$$

Physics heuristic

- We want the LDP of $\mathcal{H}_\epsilon(2,0) \approx -\lambda$ as $\lambda \to \infty$.
  - Scale $h(t,x) \to \lambda h(t, \lambda^{-\frac{1}{2}} x), \rho \to \lambda \rho(t, \lambda^{-\frac{1}{2}} x)$
- Under this scaling,

$$\partial_t h - \frac{1}{2\lambda}\partial_{xx} h - \frac{1}{2}(\partial_x h)^2 = \rho$$

$$\partial_t \rho + \frac{1}{2\lambda}\partial_{xx}\rho = \partial_x\left(\rho \partial_x h\right)$$

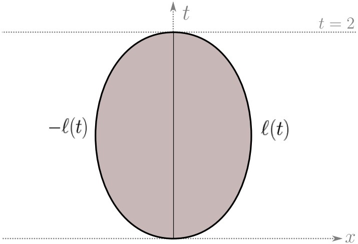
Physics heuristic

- Remove the viscosity as $\lambda \to \infty$, we have

$$\partial_t h - \frac{1}{2}(\partial_x h)^2 = \rho$$
$$\partial_t \rho = \partial_x \left(\rho \partial_x h\right)$$

- Solve the PDEs, one gets

$$\rho_*(t,x) = -\frac{1}{2\pi\ell(t)}\left(1 - \frac{x^2}{\ell(t)^2}\right)\mathbf{1}_{\{|x|\le \ell(t)\}}, \qquad \frac{1}{2}\|\rho_*\|^2 = \frac{4}{15\pi}$$



support of ρ_*

- This suggests that $\phi_-(\lambda) \approx \frac{4}{15\pi}\lambda^{\frac{5}{2}}$, $\lambda \to \infty$.

One major problem

The link between the large deviations of the KPZ equation and weak noise PDE is not rigorous.

Asymptotics of the rate function

- Scale $\rho \to \lambda\rho(t, \lambda^{-\frac{1}{2}}x)$, then

$$\phi(-\lambda) = \lambda^{\frac{5}{2}} \inf\left\{\frac{1}{2}\|\rho\|^2 : \mathcal{S}_\lambda(\rho) \le -1\right\}$$

where $\mathcal{S}_\lambda(\rho) = \lambda^{-1}\log\mathbb{E}\Big[\exp\big(\lambda\int_0^2 \rho(s, \lambda^{-\frac{1}{2}}B_{\mathsf{b}}(s))ds\big)\Big]$

- Only need to show

$$\lim_{\lambda\to\infty} \inf\left\{\frac{1}{2}\|\rho\|^2 : \mathcal{S}_\lambda(\rho) \le -1\right\} = \frac{4}{15\pi}$$

- We have the input from the physics heuristic

$$\rho_*(t,x) = -\frac{1}{2\pi\ell(t)}\Big(1 - \frac{x^2}{\ell(t)^2}\Big)\mathbf{1}_{\{|x|\le\ell(t)\}}, \qquad \frac{1}{2}\|\rho_*\|^2 = \frac{4}{15\pi}$$

  - Step 1: $\lim_{\lambda\to\infty}\mathcal{S}_\lambda(\rho_*) = -1$
  - Step 2: As $\lambda\to\infty$, prove ρ_* is the minimizer.

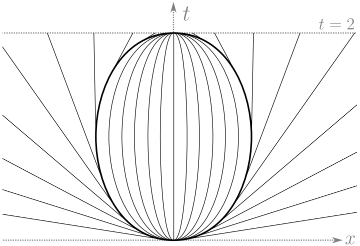
Step 1: Varadhan's lemma

- By Varadhan's lemma

$$\lim_{\lambda\to\infty} \mathcal{S}_\lambda(\rho_*) = \lim_{\lambda\to\infty} \lambda^{-1} \log \mathbb{E}_{0\to 0}\Big[\exp\big(\lambda \int_0^2 \rho_*(s, \lambda^{-\frac{1}{2}} B_{\mathsf{b}}(s))ds\big)\Big]$$
$$= \sup_{x(0)=x(2)=0} \Big(\int_0^2 \rho_*(s,x(s)) - \frac{1}{2}\dot{x}(s)^2 ds\Big)$$

- We solve the Euler-Lagrange equation

$$\ddot{x}(s) + \partial_x \rho_*(s, x(s)) = 0$$



Geodesic (solution) is not unique.

$$x(t) = \alpha\ell(t), \alpha \in [-1,1]$$
$$\int_0^2 \rho_*(s,x(s)) - \frac{1}{2}\dot{x}(s)^2 ds = -1$$

This family of geodesics is useful for Step 2.

Step 2: Girsanov's theorem

- We consider $\rho \in L^2([0,2] \times \mathbb{R})$ such that $\mathcal{S}_\lambda(\rho) = \lambda^{-1} \log \mathbb{E}\Big[\exp\big(\lambda \int_0^2 \rho(s, \lambda^{-\frac{1}{2}} B_{\mathsf{b}}(s))ds\big)\Big] \le -1$
- Jensen's inequality $\mathbb{E}\big[\int_0^2 \rho(s, \lambda^{-\frac{1}{2}} B_{\mathsf{b}}(s))ds\big] \le -1$
- Girsanov's theorem $B_b(s) \to B_b(s) + \lambda^{\frac{1}{2}} x(s)$ and Jensen's inequality,

$$\mathbb{E}\Big[\int_0^2 \rho\big(s, \lambda^{-\frac{1}{2}} B_{\mathsf{b}}(s) + x(s)\big)ds\Big] - \frac{1}{2}\int_0^2 \dot{x}(s)^2 ds \le -1$$

- Take $\lambda = \infty$ and $x(s) = \alpha\ell(s)$, we get for $\alpha \in [-1,1]$,

$$\int_0^2 \rho(s, \alpha\ell(s))ds - \frac{1}{2}\alpha^2 \int_0^2 \dot{\ell}(s)^2 ds \le -1.$$

Recover ρ_*

- For any $\alpha \in [-1, 1]$,

$$\int_0^2 \rho(s, \alpha\ell(s))ds - \rho_*(s, \alpha\ell(s))ds \leq 0.$$

- Multiplying above by $(1-\alpha^2)_+$ and integrate against α,

$$\int\int_0^2 (1-\alpha^2)_+ \big(\rho(s,\alpha\ell(s)) - \rho_*(s,\alpha\ell(s))\big) ds d\alpha \leq 0$$

- Change of variable $\alpha\ell(s) \to x$, we get

$$\int\int_0^2 \underbrace{\frac{1}{\ell(s)}(1-\frac{x^2}{\ell^2(s)})}_{-2\pi\rho_*(s,x)}(\rho(s,x) - \rho_*(s,x)) ds dx \leq 0$$

- This implies that $\|\rho\| \geq \|\rho_*\|$.

Thank you