Markov duality for stochastic six vertex model

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Markov duality

Definition

Let $X(t), Y(t)$ be two time-homogeneous Markov processes on state spaces $\mathbf{X}, \mathbf{Y}$. We say $X(t)$ and $Y(t)$ are dual with respect to the function $D : \mathbf{X} \times \mathbf{Y} \to \mathbb{R}$, if

$$\mathbb{E}_x[D(X(t), y)] = \mathbb{E}_y[D(x, Y(t))]$$

for any $x \in \mathbf{X}$, $y \in \mathbf{Y}$ and $t \geq 0$.

S6V model

- Studied by [Gwa-Sphon 92]. Tiling model with six possible vertex configurations.

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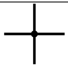
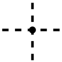
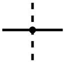
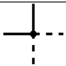
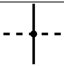
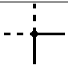
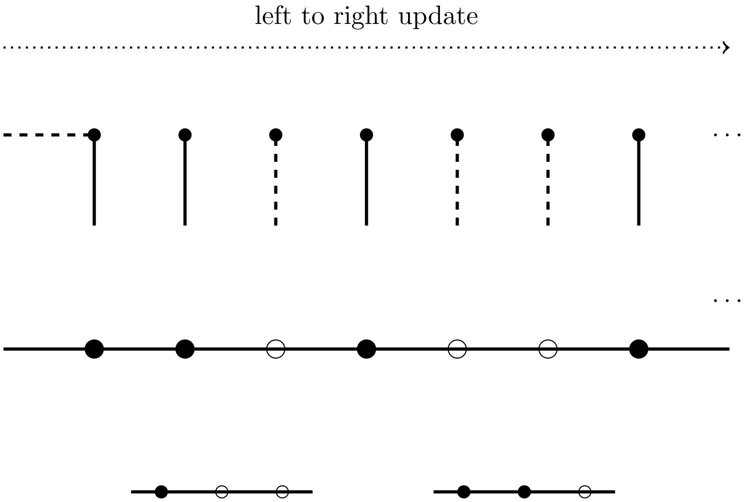
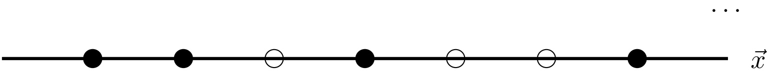
Configuration						
Weight	1	1	b_2	$1 - b_2$	b_1	$1 - b_1$

- We view it as an interacting particle system on $\mathbb{Z}$. Two parameters $0 < b_1, b_2 < 1$.

Update rule



Height function



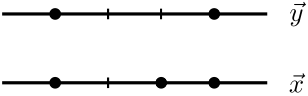
Height function $N_y(\vec{x}) = \#\{\text{particles which stay on the left or at } y\}$

Result

Theorem (L. 19)

Set $q = \frac{b_1}{b_2}$. *The S6V model* $X(t) = \big(x_1(t) < \cdots < x_k(t)\big)$ *and the space reversed S6V model* $Y(t) = \big(y_1(t) > \cdots > y_m(t)\big)$ *are dual w.r.t.*

$$H(\vec{x}, \vec{y}) = \prod_{i=1}^{m} q^{-N_{y_i}(\vec{x})} \mathbf{1}_{y_i}(\vec{x}).$$



Questions

- **Why we expect that this is a duality function?**
- How do we prove this duality?
- What is the use of this duality?

ASEP

- The ASEP on $\mathbb{Z}$.

S6V model to ASEP

- Let $b_1 \to \varepsilon b_1$, $b_2 \to \varepsilon b_2$. Scale time $t \to \varepsilon^{-1} t$ and consider a moving frame with speed 1, [Borodin-Corwin-Gorin 16], [Aggarwal 17] showed that $X(\varepsilon^{-1} t) - \varepsilon^{-1} t$ converges to ASEP with left jump rate b_1 and right jump rate b_2.

ASEP duality

- Let $X(t) = \big(x_1(t) < \cdots < x_k(t)\big)$ denotes ASEP with left jump rate ℓ and right jump rate r. $Y(t) = \big(y_1(t) > \cdots > y_m(t)\big)$ be the ASEP with left jump rate r and right jump rate ℓ. Define $q = \frac{\ell}{r}$.

- [Schütz 97] proved that $X(t)$ and $Y(t)$ are dual w.r.t.

$$H(\vec{x}, \vec{y}) = \prod_{i=1}^{m} q^{-N_{y_i}(\vec{x})} \mathbf{1}_{y_i}(\vec{x}).$$

- [Borodin-Corwin-Sasamoto 14] and [Corwin-Petrov 16] showed that both ASEP and the S6V model are self-dual w.r.t.

$$G(\vec{x}, \vec{y}) = \prod_{i=1}^{m} q^{-N_{y_i}(\vec{x})}.$$

- We wonder whether H is a duality function for the S6V model.

Result

Theorem (L. 19)

The S6V model $X(t) = \big(x_1(t) < \cdots < x_k(t)\big)$ *and the space reversed S6V model* $Y(t) = \big(y_1(t) > \cdots > y_m(t)\big)$ *are dual w.r.t.*

$$H(\vec{x}, \vec{y}) = \prod_{i=1}^{m} q^{-N_{y_i}(\vec{x})} \mathbf{1}_{y_i}(\vec{x}).$$

- Why we expect that this is a duality function? ✓
- **How do we prove this duality?**

Proof idea

- It suffices to show

$$\mathbb{E}_{\vec{x}}[H(X(1),\vec{y})] = \mathbb{E}_{\vec{y}}[H(\vec{x},Y(1))]. \qquad (*)$$

- We prove $(*)$ by using an inductive argument over the particle number.

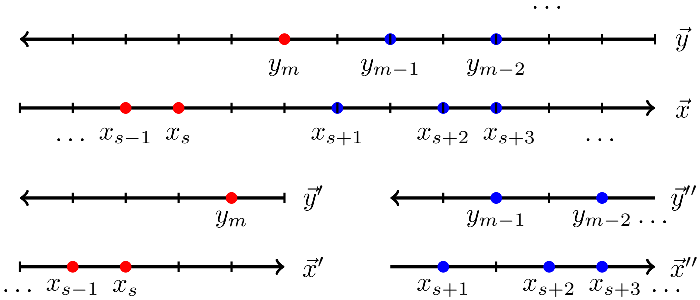
- Assume that $\vec{x}$ has k particles and $\vec{y}$ has m particles.

  Induction basis: $k = 1$ or $m = 1$.

  Induction argument: If $(*)$ holds for all $k' + m' < k + m$, show $(*)$ also holds for (k, m)

Inductive proof

- If $y_m \notin \vec{x}$

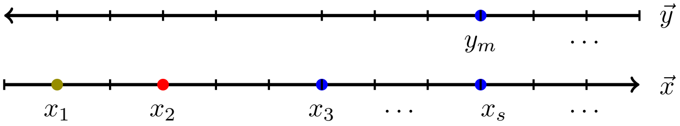


- We have

$$\mathbb{E}_{\vec{x}}\left[H(X(1),\vec{y})\right] = q^{-s(m-1)}\mathbb{E}_{\vec{x}'}\left[H(X(1),\vec{y}')\right]\mathbb{E}_{\vec{x}''}\left[H(X(1),\vec{y}'')\right],$$
$$\mathbb{E}_{\vec{y}}\left[H(\vec{x},Y(1))\right] = q^{-s(m-1)}\mathbb{E}_{\vec{y}'}\left[H(\vec{x}',Y(1))\right]\mathbb{E}_{\vec{y}''}\left[H(\vec{x}'',Y(1))\right].$$

Inductive proof

- If $y_m \in \vec{x}$ (assume that $y_m > x_2$)



- $\vec{x}' = \vec{x} - \{x_1\}$ and $\vec{x}'' = \vec{x} - \{x_1, x_2\}$.

$$L_1 = \mathbb{E}_{\vec{x}'}[H(X(1), \vec{y})], \qquad L_2 = \mathbb{E}_{\vec{x}''}[H(X(1), \vec{y})].$$
$$R_1 = \mathbb{E}_{\vec{y}}[H(\vec{x}', Y(1))], \qquad R_2 = \mathbb{E}_{\vec{y}}[H(\vec{x}'', Y(1))].$$

We have

$$\mathbb{E}_{\vec{x}}[H(X(1), \vec{y})] = f(L_1, L_2),$$
$$\mathbb{E}_{\vec{y}}[H(\vec{x}, Y(1))] = f(R_1, R_2).$$

Result

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$$H(\vec{x}, \vec{y}) = \prod_{i=1}^{m} q^{-N_{y_i}(\vec{x})} \mathbf{1}_{y_i}(\vec{x}).$$

- Why we expect that this is a duality function? ✓
- How do we prove this duality? ✓
- **What is the use of this duality?**

KPZ limit

- [Corwin-Ghosal-Shen-Tsai 18] We scale $q = e^{\sqrt{\varepsilon}}$ and fix b_1, under near equilibrium initial condition (density ρ),

$$\sqrt{\varepsilon}\Big(N(\varepsilon^{-2}t, \varepsilon^{-1}x + \varepsilon^{-2}\mu_\varepsilon t) - \rho(\varepsilon^{-1}x + \varepsilon^{-2}\mu_\varepsilon t)\Big) - \varepsilon^{-2} t \log \lambda_\varepsilon \Rightarrow h,$$

  where h solves the KPZ equation

$$\partial_t h(t,x) = \partial_{xx} h(t,x) + (\partial_x h(t,x))^2 + \xi(t,x).$$

- The solution to the KPZ equation is the log of the Stochastic heat equation: $h = \log Z$, where

$$\partial_t \mathcal{Z}(t,x) = \partial_{xx}\mathcal{Z}(t,x) + \xi(t,x)\mathcal{Z}(t,x).$$

KPZ limit

- One particle duality [Corwin-Petrov 16] shows that

$$\mathbb{E}\Big[q^{-N(t+1,x)} \,\big|\, \mathcal{F}(t)\Big] = \sum_{y} p(x-y) q^{-N(t,x)}.$$

Let $Z(t,x)$ denote a version of $q^{-N(t,x)}$, we have the discrete SHE

$$dZ(t,x) = \big(\widetilde{p} \star Z(t)\big)(x) + M(t,x).$$

Using duality in [Corwin-Petrov 16] and [L. 19] to prove that the discrete SHE converges to its continuum

$$\partial_t \mathcal{Z}(t,x) = \partial_{xx} \mathcal{Z}(t,x) + \xi(t,x)\mathcal{Z}(t,x).$$

KPZ limit

For $x_1 \le x_2$, we have

$$\mathbb{E}\big[M(t,x_1)M(t,x_2)\big|\mathcal{F}(t)\big] = \Theta_1(t,x_1)\Theta_2(t,x_1),$$

where

$$\varepsilon^{-\frac{1}{2}}\Theta_1(t,x) = c_0 Z(t,x) + \sum_{i=1}^{\infty} c_i \varepsilon^{-\frac{1}{2}} \nabla Z(t,x-i),$$

$$\varepsilon^{-\frac{1}{2}}\Theta_2(t,x) := c_0' Z(t,x) + \sum_{i=1}^{\infty} c_i' \varepsilon^{-\frac{1}{2}} \nabla Z(t,x-i).$$

The Markov dualities imply self-averaging in time.

Thank you!