KPZ equation with a small noise, deep upper tail and limit shape

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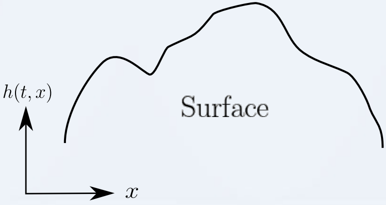
The KPZ equation

- Introduced by [Kardar-Parisi-Zhang 86]

$$\partial_t h = \frac{1}{2}\partial_{xx} h + \frac{1}{2}(\partial_x h)^2 + \xi,$$

ξ is the space-time white noise.



- The solution theory is ill-posed.

The SHE

- Look at the stochastic heat equation (SHE)

$$\partial_t Z = \frac{1}{2}\partial_{xx} Z + \xi Z.$$

  and define $h := \log Z$.

- Mild solution of the SHE

$$Z(t,x) = \int_{\mathbb{R}} p(t,x-y)Z(0,y)dy + \int_0^t \int_{\mathbb{R}} p(t-s,x-y)Z(s,y)\xi(s,y)dsdy.$$

KPZ equation

- [Mueller 91] postivity of the solution to the SHE.

- Dirac-Delta initial data
$$\partial_t Z = \frac{1}{2}\partial_{xx} Z + \xi Z, \qquad Z(0,\cdot) = \delta(\cdot).$$
[Moreno-Flores 14] $Z(t,x)$ is positive for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$.

- Define $h = \log Z$ to be the solution to the KPZ equation starting from the **narrow wedge initial data.**

Long time fluctuation

- [Amir-Corwin-Quastel 11]

$$\frac{h(2t,0)+\frac{t}{12}}{t^{\frac{1}{3}}} \Rightarrow F_{\mathrm{GUE}}, \qquad t \to \infty.$$

Long time large deviations

- **Large deviation principle (LDP)** [Das-Tsai 19]: For $\lambda \geq 0$,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(h(2t,0) + \frac{t}{12} \geq \lambda t\Big) = -\frac{4}{3}\lambda^{\frac{3}{2}}.$$

  Generalized to the general initial data by [Ghosal-Lin 20].

- [Tsai 18, Cafasso-Claeys 19] For $\lambda \geq 0$,

$$\lim_{t\to\infty} \frac{1}{t^2} \log \mathbb{P}\Big(h(2t,0) + \frac{t}{12} \leq -\lambda t\Big) = -\phi(\lambda),$$

  where ϕ is explicit. $\lim_{\lambda\to 0} \frac{\phi(\lambda)}{\lambda^3} = \frac{1}{12}$, $\lim_{\lambda\to\infty} \frac{\phi(\lambda)}{\lambda^{\frac{5}{2}}} = \frac{4}{15}$.

Short time behavior

[Amir-Corwin-Quastel 11]

$$\frac{h(2t,0)+\log\sqrt{4\pi t}}{t^{\frac{1}{4}}} \Rightarrow \mathcal{N}\Big(0,\sqrt{\tfrac{\pi}{2}}\Big), \qquad t\to 0.$$

What about short time large deviations?

Theorem (L.-Tsai 20)

There exist $\Phi:\mathbb{R}\to\mathbb{R}_{\geq 0}$ *s.t. for any* $\lambda\geq 0$,

$$\lim_{t\to 0}\sqrt{t}\log\mathbb{P}\big(h(2t,0)+\log(\sqrt{4\pi t})\geq\lambda\big) = -\Phi(\lambda),$$

$$\lim_{t\to 0}\sqrt{t}\log\mathbb{P}\big(h(2t,0)+\log(\sqrt{4\pi t})\leq-\lambda\big) = -\Phi(-\lambda)$$

In addition,

$$\lim_{\lambda\to 0}\frac{\Phi(\lambda)}{\lambda^2}=\frac{1}{\sqrt{2\pi}}, \qquad \lim_{\lambda\to\infty}\frac{\Phi(-\lambda)}{\lambda^{\frac{5}{2}}}=\frac{4}{15\pi}$$

An explicit form of Φ is predicted in [Le Doussal-Majumdar-Rosso-Schehr 16].

Weak noise scaling

- Equivalent to the short time regime.
- Consider
$$\partial_t h_\varepsilon = \frac{1}{2}\partial_{xx} h_\varepsilon + \frac{1}{2}(\partial_x h_\varepsilon)^2 + \sqrt{\varepsilon}\xi.$$
- $h_\varepsilon(\bullet,\bullet) \stackrel{d}{=} h(\varepsilon^2\bullet, \varepsilon\bullet) + \log\varepsilon$.
- The result of [L.-Tsai 20] becomes: For $\lambda \geq 0$,
$$\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\big(h_\varepsilon(2,0) + \log(\sqrt{4\pi}) \geq \lambda\big) = -\Phi(\lambda)$$
$$\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\big(h_\varepsilon(2,0) + \log(\sqrt{4\pi}) \leq -\lambda\big) = -\Phi(-\lambda)$$
and $\lim_{\lambda\to 0} \frac{\Phi(\lambda)}{\lambda^2} = \frac{1}{\sqrt{2\pi}}, \lim_{\lambda\to\infty} \frac{\Phi(-\lambda)}{\lambda^{\frac{5}{2}}} = \frac{4}{15\pi}$.

Deep Upper tail and limit shape

Theorem (Lamarre-L.-Tsai 21)

We have $\lim_{\lambda\to\infty} \frac{\Phi(\lambda)}{\lambda^{\frac{3}{2}}} = \frac{4}{3}$.

Theorem (Lamarre-L.-Tsai 21)

Fix $\delta > 0$. *Define* $h_{\varepsilon,\lambda}(t,x) = h_\varepsilon(t, \lambda^{\frac{1}{2}} x)$, *we have*

$$\lim_{\lambda\to\infty} \lim_{\varepsilon\to 0} \mathbb{P}\Big(\|h_{\varepsilon,\lambda} - \mathsf{h}_*\|_{L^\infty([\delta,2]\times[-\delta^{-1},\delta^{-1}])} < \delta \,\Big|\, h_\varepsilon(2,0) + \log\sqrt{4\pi} \geq \lambda \Big) = 1,$$

where

$$\mathsf{h}_*(t,x) = \begin{cases} \frac{t}{2} - |x|, & |x| \leq t, \\ -\frac{x^2}{2t} & |x| \geq t. \end{cases}$$

KPZ equation with a small noise

- We study the large deviation of the path h_ε on $C([\eta, T] \times \mathbb{R})$.

- Go to the SHE,

$$\partial_t Z_\varepsilon = \frac{1}{2}\partial_{xx} Z_\varepsilon + \sqrt{\varepsilon}\xi Z_\varepsilon.$$

  Recall $h_\varepsilon := \log Z_\varepsilon$, it suffices to study Z_ε.

Freidlin Wentzell LDP

$\mathbb{P}(\sqrt{\varepsilon}\xi \approx \rho) \approx \exp\big(-\frac{1}{2}\varepsilon^{-1}\|\rho\|_{L^2}^2\big)$. Consider

$$\partial_t \mathsf{Z} = \frac{1}{2}\partial_{xx}\mathsf{Z} + \rho\mathsf{Z}, \qquad \mathsf{Z}(0,\cdot) = \delta(\cdot).$$

Denote the solution by $\mathsf{Z}(\rho)$. By the Feynman-Kac formula,

$$\mathsf{Z}(\rho)(t,x) = \mathbb{E}_{0\to x}\Big[\exp\Big(\int_0^t \rho(s, B_b(s))ds\Big)\Big]p(t,x).$$

Theorem (L.-Tsai 20)

Fix $\eta > 0$. $\{Z_\varepsilon(\bullet,\bullet)\}_{\varepsilon\in(0,1)} \subseteq C([\eta,2]\times\mathbb{R})$ satisfies a LDP with speed ε^{-1} and a good rate function

$$I(f) = \inf\Big\{\frac{1}{2}\|\rho\|^2_{L^2([0,2]\times\mathbb{R})} : \mathsf{Z}(\rho) = f, \rho \in L^2([0,2]\times\mathbb{R})\Big\}.$$

The one-point rate function

- By contraction principle, for $\lambda \geq 0$,

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{P}\Big(h_\varepsilon(2,0) + \log \sqrt{4\pi} \geq \lambda\Big) = -\Phi(\lambda),$$

  where

$$\Phi(\lambda) = \inf\left\{\frac{1}{2}\|\rho\|^2_{L^2([0,2]\times\mathbb{R})} : \mathsf{Z}(\rho; 2, 0) \geq \frac{1}{\sqrt{4\pi}} e^\lambda\right\}.$$

An important observable

- Let
$$\widetilde{\mathcal{K}}_\lambda = \arg\inf \left\{ \tfrac{1}{2}\|\rho\|^2_{L^2([0,2]\times\mathbb{R})} : \mathsf{Z}(\rho;2,0) \geq \frac{1}{\sqrt{4\pi}} e^{\lambda} \right\}.$$
- $\widetilde{\mathcal{K}}_\lambda$ represents the $\varepsilon \to 0$ limit shape of $\sqrt{\varepsilon}\xi$, if we force $Z_\varepsilon(2,0) \geq \frac{1}{\sqrt{4\pi}} e^{\lambda}$.
- Let $\widetilde{\mathcal{K}}_\lambda = \{\lambda\rho(\lambda\bullet, \lambda^{\frac{1}{2}}\bullet) : \rho \in \mathcal{K}_\lambda\}$, where
$$\mathcal{K}_\lambda = \arg\inf \left\{ \tfrac{1}{2\lambda}\|\rho\|^2_{L^2([0,2\lambda]\times\mathbb{R})} : \mathsf{Z}(\rho;2\lambda,0) \geq \frac{1}{\sqrt{4\pi\lambda}} e^{\lambda} \right\}.$$
Note that
$$\Phi(\lambda) = \lambda^{\frac{3}{2}} \inf \left\{ \tfrac{1}{2\lambda}\|\rho\|^2_{L^2([0,2\lambda]\times\mathbb{R})} : \mathsf{Z}(\rho;2\lambda,0) \geq \frac{1}{\sqrt{4\pi\lambda}} e^{\lambda} \right\}.$$
- How does $\mathcal{K}_\lambda$ behave as $\lambda \to \infty$?

Heuristic

- **Assume** ρ is **time-independent** and write $\rho(t,\bullet) = \varphi(\bullet)$. Then

$$\mathcal{K}_\lambda = \arg\inf \Big\{ \|\varphi\|^2_{L^2(\mathbb{R})} : \mathsf{Z}(\varphi; 2\lambda, 0) \geq \frac{1}{\sqrt{4\pi\lambda}} e^\lambda \Big\}.$$

- View the PDE as $\frac{d}{dt}\mathsf{Z}(t) = A^\varphi \mathsf{Z}(t)$, where $A^\varphi = \frac{1}{2}\partial_{xx} + \varphi$. $\mathsf{Z}(\varphi; 2\lambda, 0)$ should grow as $\exp(2\lambda F(\varphi))$ as $\lambda \to \infty$, where

$$F(\varphi) = \sup \Big\{ \int_{\mathbb{R}} \varphi g^2 - \frac{1}{2} g'^2 : g \in H^1(\mathbb{R}) \text{ and } \|g\|_{L^2(\mathbb{R})} = 1 \Big\}.$$

- Heuristically, $\mathsf{Z}(\varphi; 2\lambda, 0) \sim \exp(2\lambda F(\varphi))$, as $\lambda \to \infty$, $\mathcal{K}_\lambda$ **approximates**

$$\arg\inf \{ \|\varphi\|^2_{L^2(\mathbb{R})} : F(\varphi) \geq \frac{1}{2} \}.$$

A tight bound of F and its optimizer

Recall that

$$F(\varphi) = \sup\left\{ \int_{\mathbb{R}} \varphi g^2 - \frac{1}{2} g'^2 : g \in H^1(\mathbb{R}) \text{ and } \|g\|_{L^2(\mathbb{R})} = 1 \right\}.$$

Lemma

We have

$$F(\varphi) \leq \frac{1}{2}\left(\frac{3}{4}\right)^{\frac{2}{3}} \|\varphi\|_{L^2(\mathbb{R})}^{\frac{4}{3}}.$$

The inequality becomes an equality if and only if $\varphi = \alpha^2 \mathrm{sech}(\alpha(\bullet - v))$.

We have

$$\arg\inf\left\{ \|\varphi\|_{L^2(\mathbb{R})}^2 : F(\varphi) \geq \frac{1}{2} \right\} = \{\mathrm{sech}^2(\bullet - v)\}_{v \in \mathbb{R}}$$

Heuristic

- The minimizers should be symmetry. So $v = 0$. Let $\rho_*(t,x) = \mathrm{sech}^2 x$.
- Condition on $h_\varepsilon(2,0) + \log\sqrt{4\pi} \geq \lambda$, $\sqrt{\varepsilon}\xi$ concentrate around $\widetilde{\mathcal{K}}_\lambda$. $h_{\varepsilon,\lambda}(t,x)$ concentrates around
$$\lambda^{-1}\log \mathsf{Z}(\widetilde{\mathcal{K}}_\lambda; t, \lambda^{\frac{1}{2}}x) = \lambda^{-1}\log \lambda^{\frac{1}{2}}\mathsf{Z}(\mathcal{K}_\lambda; \lambda t, \lambda x).$$
- Heuristically, $\mathcal{K}_\lambda$ converges to ρ_* as $\lambda \to \infty$. Hence, $h_{\varepsilon,\lambda}(t,x)$ concentrates around
$$\lambda^{-1}\log \lambda^{\frac{1}{2}}\mathsf{Z}(\rho_*; \lambda t, \lambda x).$$

The limit shape

- As $\varepsilon \to 0$, $h_{\varepsilon,\lambda}(t,x)$ concentrates around

$$\lambda^{-1} \log \lambda^{\frac{1}{2}} \mathsf{Z}(\rho_*; \lambda t; \lambda x)$$
$$= \lambda^{-1} \log \mathbb{E}_{\lambda x \to 0} \left[\exp\left(\int_0^{\lambda t} \rho_*(B_b(s) ds) \right) \right] - \frac{x^2}{2t} - \lambda^{-1} \log \sqrt{4\pi}$$

- The $\lambda \to \infty$ asymptotic of above equals

$$\mathsf{h}_*(t,x) = \begin{cases} \frac{t}{2} - |x|, & |x| \leq t, \\ -\frac{x^2}{2t} & |x| \geq t. \end{cases}$$

Technical Challenges

- Characterization of $\mathcal{K}_\lambda$.
- How can we prove that as $\lambda \to \infty$, $\mathcal{K}_\lambda$ should converge to ρ_*.
- $\lambda^{-1} \log \lambda^{\frac{1}{2}} \mathsf{Z}(\mathcal{K}_\lambda; \lambda t, \lambda x)$. concentrates uniformly around $\lambda^{-1} \log \mathsf{Z}(\rho_*; \lambda t, \lambda x)$.
- The asymptotic of exponential of a Brownian bridge.

Property of $\mathcal{K}_\lambda$

- Recall that

$$\mathcal{K}_\lambda = \arg\inf\left\{\tfrac{1}{2\lambda}\|\rho\|^2_{L^2([0,2\lambda]\times\mathbb{R})} : \mathsf{Z}(\rho; 2\lambda, 0) \geq \frac{1}{\sqrt{4\pi\lambda}}e^\lambda\right\}.$$

- Introduce a abstract Wiener space $(\mathcal{B}, \mu)$ such that $L^2([0, 2\lambda] \times \mathbb{R})$ is the Cameron-Martin space. We have

$$\mathsf{Z} : \rho \to \mathsf{Z}(\rho; 2\lambda, 0).$$

  is continuous. Moreover, $\{\|\rho\|^2_{L^2([0,2\lambda]\times\mathbb{R})} \leq r\}$ forms a **compact** set in $\mathcal{B}$. This guarantees that $\mathcal{K}_\lambda$ is non-empty.

- $\mathcal{K}_\lambda$ only contains non-negative and symmetric decreasing ρ.

Property of $\mathcal{K}_\lambda$

We have the following result for the exponential integral of the Brownian bridge.

Lemma

For φ continuous and bounded,

$$\lim_{\lambda\to\infty} \lambda^{-1} \log \mathbb{E}_{0\to 0}\left[\exp\left(\int_0^\lambda \varphi(B_b(s))\right)\right] = F(\varphi).$$

As a consequence, for $\rho_*(x) = \operatorname{sech}^2 x$,

$$\mathsf{Z}(\rho_*; 2\lambda, 0) = \mathbb{E}_{0\to 0}\left[\exp\left(\int_0^{2\lambda} \rho_*(B_b(s))ds\right)\right] \sim e^\lambda.$$

For $\rho \in \mathcal{K}_\lambda$, $\frac{1}{2\lambda}\|\rho\|^2_{L^2([0,2\lambda]\times\mathbb{R})} \leq \frac{4}{3} + o_\lambda(1)$.

$\mathcal{K}_\lambda$ converges to ρ_*

- How to prove that $\mathcal{K}_\lambda$ converges to ρ_* as $\lambda \to \infty$.
- What does $\mathsf{Z}(\rho; 2\lambda, 0) \geq \frac{1}{\sqrt{4\pi\lambda}} e^{\lambda}$ tell about the L^2 norm of ρ.
- Define the kernel $P(\rho; (s, y) \to (t, x))$ to be the solution $\mathsf{Z}(t, x)$ of

$$\partial_t \mathsf{Z}(t, x) = \frac{1}{2} \partial_{xx} \mathsf{Z}(t, x) + \rho(t, x) \mathsf{Z}(t, x), \qquad \mathsf{Z}(s, \bullet) = \delta_y(\bullet).$$

- Define the operator $P(\rho; s \to t) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ such that

$$(P(\rho; s \to t) f)(x) = \int P(\rho; (s, y) \to (t, x)) f(y) dy.$$

$\mathcal{K}_\lambda$ converges to ρ_*

Semigroup Property:

$$\begin{aligned}\mathsf{Z}(\rho;2\lambda,0) &= P(\rho;(2\lambda-1,\bullet)\to(2\lambda,0))P(\rho;1\to 2\lambda-1)P((0,0)\to(1,\bullet))\\ &\le \|P(\rho;(2\lambda-1,\bullet)\to(2\lambda,0))\|_{L^2(\mathbb{R})}\|P(\rho;1\to 2\lambda-1)\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})}\\ &\quad\times\|P((0,0)\to(1,\bullet))\|_{L^2(\mathbb{R})}\end{aligned}$$

Lemma

We have

$$\|P(\rho;(0,0)\to(1,\bullet))\|_{L^2(\mathbb{R})}\le C\exp\left(a^2C+\frac{1}{a}\|\rho\|^2_{L^2([0,1]\times\mathbb{R})}\right)$$

$$\|P(\rho;(2\lambda-1,\bullet)\to(2\lambda,\bullet))\|_{L^2(\mathbb{R})}\le C\exp\left(a^2C+\frac{1}{a}\|\rho\|^2_{L^2([2\lambda-1,2\lambda]\times\mathbb{R})}\right)$$

$\mathcal{K}_\lambda$ converges to ρ_*

Proposition

For non-negative $\rho \in L^2([s,t]\times\mathbb{R})$,

$$\|P(\rho; s \to t)\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \le \exp\left(\int_s^t F(\rho(r,\cdot))dr\right).$$

Assume f, ρ are C_c^∞. Let $\mathsf{Z}(t,x) = (P(s \to t)f)(x)$,

$$\partial_r \mathsf{Z}(r,x) = \frac{1}{2}\partial_{xx}\mathsf{Z}(r,x) + \rho(r,x)\mathsf{Z}(r,x).$$

Multiply both sides by $\mathsf{Z}(r,x)$ and integrate in x,

$$\frac{1}{2}\partial_r \|\mathsf{Z}(r,\cdot)\|^2_{L^2(\mathbb{R})} \le F(\rho(r,\cdot))\|\mathsf{Z}(r,\cdot)\|^2_{L^2(\mathbb{R})}$$

Integrate in r, $\|\mathsf{Z}(t,\cdot)\|^2_{L^2(\mathbb{R})} \le \exp\left(2\int_s^t F(\rho(r,\cdot))dr\right)\|f\|^2_{L^2(\mathbb{R})}$.

General result follows by approximation and monotonicity.

$\mathcal{K}_\lambda$ converges to ρ_*

- If $\rho \in \mathcal{K}_\lambda$,

$$\frac{1}{2\lambda}\|\rho\|^2_{L^2([0,2\lambda]\times\mathbb{R})} \le \frac{4}{3} + o_\lambda(1).$$

- We have

$$\frac{1}{\sqrt{4\pi\lambda}}e^\lambda \le Z(\rho;2\lambda,0) \le C\exp\left(\int_0^{2\lambda} F(\rho(r,\bullet))dr\right)$$
$$\le C\exp\left(\int_0^{2\lambda} \frac{1}{2}\left(\frac{3}{4}\right)^{\frac{2}{3}} \|\rho(r,\bullet)\|^{\frac{4}{3}}_{L^2(\mathbb{R})} dr\right)$$
$$\le C\exp\left(\int_0^{2\lambda} \frac{1}{4}\left(\|\rho(r,\bullet)\|^2_{L^2(\mathbb{R})} + \frac{2}{3}\right)dr\right) \le Ce^{\lambda + o_\lambda(1)}$$

  - $F(\rho(r,\bullet))$ can not be far from $\frac{1}{2}(\frac{3}{4})^{\frac{2}{3}}\|\rho(r,\bullet)\|^{\frac{4}{3}}_{L^2(\mathbb{R})}$.
  - $\|\rho(r,\bullet)\|^2_{L^2(\mathbb{R})}$ can not be far away from $\frac{4}{3}$.

A perturbation analysis

Lemma

Consider a sequence of symmetric decreasing function $\{\varphi_n\}_{n=1}^{\infty}$. $\|\varphi_n\|^2_{L^2(\mathbb{R})} = \frac{4}{3}$ *such that* $F(\varphi_n) \to \frac{1}{2}$, *then* $\varphi_n \to \rho_*$ *in* $L^2(\mathbb{R})$.

If $\mathsf{Z}(\rho; 2\lambda, 0) \geq \frac{1}{\sqrt{4\pi\lambda}} e^{\lambda}$ and $\frac{1}{2\lambda} \|\rho\|^2_{L^2([0,2\lambda]\times\mathbb{R})} = \frac{4}{3} + o_\lambda(1)$.

- For most $r \in [0, 2\lambda]$, $\|\rho(r, \cdot)\|^2_{L^2(\mathbb{R})}$ is close to $\frac{4}{3}$.
- For most $r \in [0, 2\lambda]$, $F(\rho(r, \cdot))$ is close to $\frac{1}{2}$.
- Hence, for most $r \in [0, 2\lambda]$, $\rho(r, \cdot)$ approximates ρ_*.

Asymptotic limit

- As $\varepsilon \to 0$, $h_{\varepsilon,\lambda}(t,x)$ concentrates around

$$\begin{aligned}&\lambda^{-1}\log\lambda^{\frac{1}{2}}\mathsf{Z}(\rho_*;\lambda t;\lambda x)\\&=\lambda^{-1}\log\mathbb{E}_{\lambda x\to 0}\Big[\exp\Big(\int_0^{\lambda t}\rho_*(B_b(s)ds)\Big)\Big]-\frac{x^2}{2t}-\lambda^{-1}\log\sqrt{4\pi}\end{aligned}$$

- Let η be the hitting time of zero. We have

$$\mathbb{E}_{\lambda x\to 0}\Big[\exp\Big(\int_0^{\lambda t}\rho_*(B_b(s)ds)\Big)\Big]\approx\mathbb{E}\Big[\exp\big(\tfrac{1}{2}(\lambda t-\eta)\big)\Big].$$

- We have $\mathbb{P}(\eta\approx\lambda s)\approx\exp(-\frac{\lambda x^2(t-s)}{2st})$. Hence the limit is

$$\sup\Big\{\frac{1}{2}(t-s)-\frac{x^2(t-s)}{2st}\Big\}=\mathsf{h}_*(t,x)+\frac{x^2}{2t}.$$

Remarks

- For the deep lower tail conditioning, we know the optimal deviation of the noise. There are some technical issues for proving the limit shape.

- The physics paper [Krajenbrink-Le Doussal 21] solves the finite λ limit shape. In the limit they discuss how the limit shape emerges. It would be interesting to prove their result rigorously.

Thank you!