A law of large numbers of the t-PNG model

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Random matrix theory Seminar

Joint work with Hindy Drillick

Ulam's problem

Consider a random permutation of $1, 2, \cdots, n$. Define ℓ_n to be the length of the longest increasing subsequence.

$$531426, \qquad \ell_6 = 3,$$
$$214536, \qquad \ell_6 = 4.$$

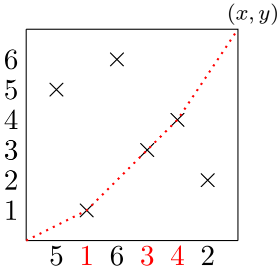
Ulam's problem corresponds to proving

$$\frac{\ell_n}{\sqrt{n}} \to c, \qquad n \to \infty.$$

and identify the constant c.

Hammersley's idea

(Hammersley 72) Put a Poisson point process on the first quadrant. Define a height function $H(x, y)$ to be the length of the longest upright path from $(0, 0)$ to (x, y).

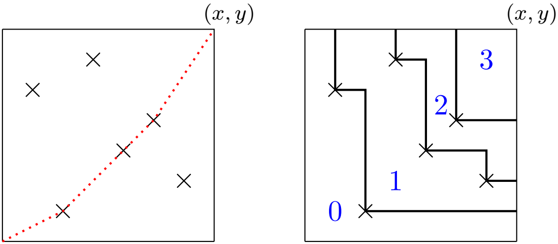


Observation

Given that there are n points in $[0, m] \times [0, m]$, $H(m, m) \stackrel{d}{=} \ell_n$.

The polynuclear growth (PNG) model

Draw lines from the points. When lines meet, they kill each other. The resulting paths play the role as level line of H.



This interpretation gives polynuclear growth (PNG) model.

Identification of constant

Ulam's problem is equivalent to proving

$$\frac{H(n,n)}{n} \to c, \qquad n \to \infty$$

which follows from Liggett's super-additive ergodic theorem.

Identification of constant: $c = 2$ by Logan-Shepp 77, Vershik-Kerov 77.

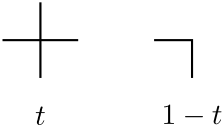
Different proofs have been given by Aldous, Cator, Diaconis, Groeneboom, Seppäläinen, ...

Using integrable method, Baik-Deift-Johansson 99 proved

$$\frac{H(n,n) - 2n}{n^{1/3}} \Rightarrow F_{\mathrm{GUE}}.$$

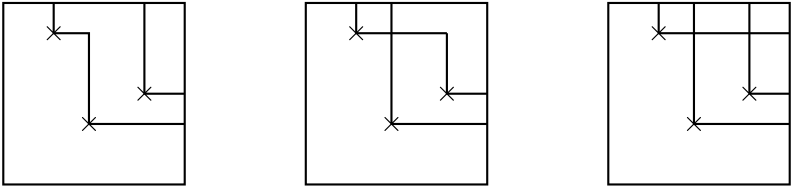
The t-PNG model

Define the t-PNG model by imposing a probability t for crossing whenever two lines meet. When $t = 0$, it reduces to the PNG model.



The model was introduced by Aggarwal-Borodin-Wheeler 21.

Given the Poisson points, here are three possible outcomes.



$\color{red}{\text{prob}} =$

The height function can be defined similarly.

The t-PNG model

Using integrable method, Aggarwal-Borodin-Wheeler 21 showed that for $t \in [0, 1)$,

$$\frac{H(n,n) - \frac{2n}{\sqrt{1-t}}}{(1-t)^{-1/6} n^{1/3}} \Rightarrow F_{\text{GUE}}, \qquad n \to \infty.$$

For $t = 1$,

$$\frac{H(n,n) - n^2}{n} \Rightarrow \text{Gaussian}.$$

Motivation

Find a probabilistic proof for a strong law of large numbers, following the spirit by Aldous, Cator, Diaconis, Groeneboom, Seppäläinen, ...

Result

Theorem (Drillick-L 22)

For $t \in [0,1)$, we have almost surely

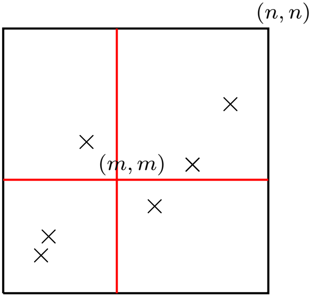
$$\frac{H(n,n)}{n} \to \frac{2}{\sqrt{1-t}}, \qquad n \to \infty.$$

The result can be easily generalized to the (simultaneous) convergence in every direction.

Liggett's super-additive ergodic theorem

For the PNG model, let $X_{m,n}$ be the length of the longest upright path from (m,m) to (n,n).

- $X_{0,n}$ has the same law as $H(n,n)$.
- $X_{0,m} + X_{m,n} \leq X_{0,n}$.
- $\{X_{nk,n(k+1)}\}_{n\geq 1}$ is ergodic.

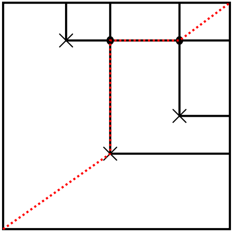


This implies $\frac{H(n,n)}{n} \to c$ a.s..

Super-additive ergodic theorem for the t-PNG model

For the t-PNG model, we introduce the α-points, which is the union of Poisson points and crossing points.

The height function is equal to the length of the longest upright paths collecting the α-points.



Define $X_{m,n}$ to be the length of longest paths for α-points from (m, m) to (n, n).

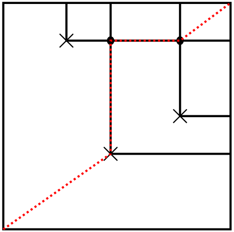
Super-additive ergodic theorem for the t-PNG model

We still have

- $X_{0,n}$ has the same law as $H(n,n)$
- $X_{0,m} + X_{m,n} \leq X_{0,n}$.

We no longer have

- $\{X_{nk,n(k+1)}\}_{n\geq 1}$ is ergodic.

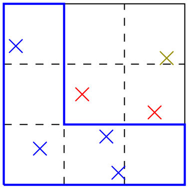


Major issue

$X_{m,n}$ depends on what happens on the left and bottom of $[m,n] \times [m,n]$.

A multicolored Model

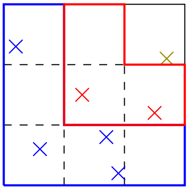
Solution: We assign each Poisson point a color.



Color -1, -2, -3 ..., smaller-valued color has higher priority.

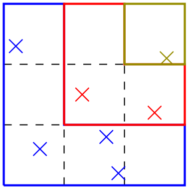
A multicolored Model

Solution: We assign each Poisson point a color.



Color -1, -2, -3 . . . , smaller-valued color has higher priority.

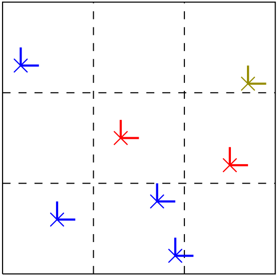
Solution: We assign each Poisson point a color.



Color -1, -2, -3 …, smaller-valued color has higher priority.

Multicolored Model

The lines inherit the color of the point they emanate from.

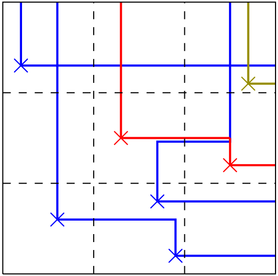


We need a rule for how different classes of lines behave when they interact with each other.

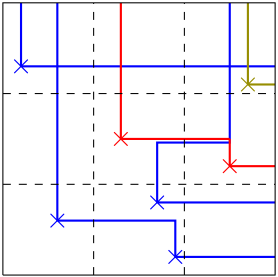
Properties of the Multicolored Model

This model needs to satisfy three important properties:

1. Color Ignorance: A line ignores lines of larger number color. e.g the red lines ignore the blue lines and the olive lines ignore the red and the blue lines.

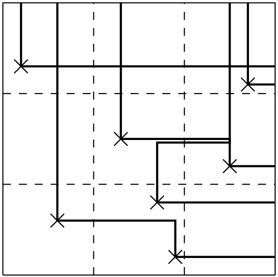


2. Color Projection: We can project this down to the single-colored t-PNG model



2. Color Projection: We can project this down to the single-colored t-PNG model

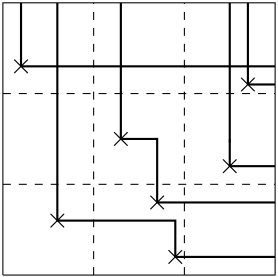
- Recolor all lines with the same color.



Properties of Multicolored Model

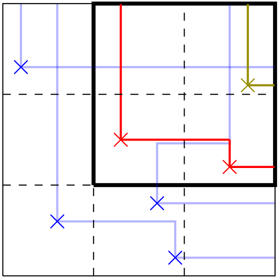
2. Color Projection: We can project this down to the single-colored t-PNG model

- Recolor all lines with the same color.
- If we have multiple lines traveling together we replace them with that number mod 2.



Properties of Multicolored Model

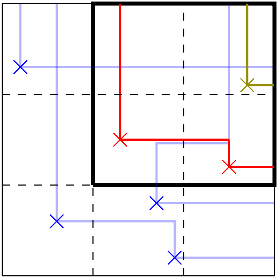
3. Monotonicity If we add lines to the box $[m, n] \times [m, n]$ entering from the bottom and left, this can only increase the height function.



Super-additive ergodic theorem

Define $X_{m,n}$ to be the height function restricted on $[m,n] \times [m,n]$, where we only look at the lines emanating from the Poisson points inside the box.

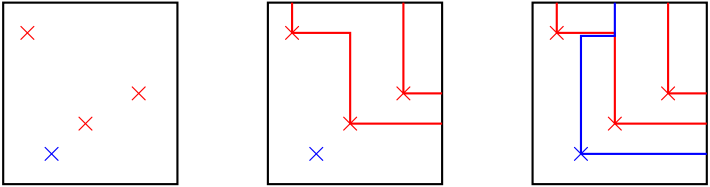
- $\{X_{nk,n(k+1)}\}_{n\geq 1}$ is ergodic.
- $X_{0,n}$ has the same law as $H(n,n)$
- $X_{0,m} + X_{m,n} \leq X_{0,n}$.



Conclusion: $\frac{H(n,n)}{n} \to c$ a.s..

About construction

The construction is motivated by the two-colored PNG model in Cator-Groeneboom 05,

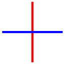
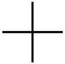
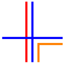
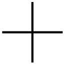


For the t-PNG model, there is extra **randomness**.

About construction

Given lines from left and bottom, how do we assign probability when they meet?

$$\mathfrak{min}(x_1, \ldots, x_n) := \begin{cases} \min(x_1, \ldots, x_n), & \text{if} \nexists\, i, j \text{ s.t. } x_i = t, x_j = 1 - t, \\ 0 & \text{else.} \end{cases}$$

Config	1-fold proj	2-fold proj	3-fold proj	prob
				$\mathfrak{min}(1, t) = t$
				$\mathfrak{min}(1, 1, 1 - t) = 1 - t$
				$\mathfrak{min}(t, 1, 1 - t) = 0$

red > blue > orange.

Some remarks

The assignment of probability follows from a sophisticated guess. In Drillick-L 22, we prove that it satisfies the desired properties by a case-by-case check.

Recently, we realized that the function $\mathfrak{min}$ can alternatively be defined using a Boolean-type product. This significantly simplifies the proof and allows us to generalize this method to other models, such as the stochastic six vertex model.

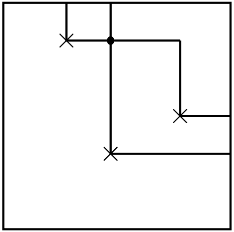
Identification of constant

The identification of c is related to a law of large numbers of the α-points.

Proposition (Drillick-L 22)

Let $N(n,n)$ be the number of α-point in $[0,n] \times [0,n]$. We have almost surely

$$\lim_{n\to\infty} \frac{N(n,n)}{n^2} = \frac{1}{1-t}.$$



The proposition implies that $c^2 = \frac{2}{1-t} + \frac{1}{2}c^2$.

Some remarks

The super-additive ergodic theorem does not rule out the possibility of $c = \infty$. We also need to show that $c < \infty$.

The t-PNG model has a stationary version. By coupling with the stationary model and Burke's property, we can obtain an upper bound for c.

Future problems

- Generalize the scope of the method to other models.
- Prove a law of large numbers (hydrodynamics limit) for the model with general boundary data.

Thank you!