San Jose Math Circle: The mystery of probability

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Preliminary

What is probability? A probability indicates the chance that an event will happen. A probability can be any number from 0 to 1. A probability of 0 means that an event will never happen. A probability of $\frac{1}{2}$ means that an event is equally likely to happen or not happen. A probability of 1 means that an event will undoubtedly happen.

The probability that event A will happen is written as $\mathbb{P}(A)$. For example, the probability of getting 1 after rolling a fair dice is $\frac{1}{6}$. The probability of getting 3 or 4 after rolling a fair dice is $\frac{2}{6} = \frac{1}{3}$. The probability that event A will not happen is $1 - \mathbb{P}(A)$. The fundamental formula for computing probability is

$$P(A) = \frac{\text{ways } A \text{ can happen}}{\text{possible outcomes}}.$$

Some warm-up problems in counting

How many ways are there to take two numbers from $\{1, 2, 3, 4, 5, 6\}$ with replacement?

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Is there a general rule behind counting the ways of taking m numbers from $\{1, 2, \ldots, n\}$ without replacement?

Some exercises

(1) (AMC 8, 2019, Problem 18) The faces of each of the two fair dice are numbered 1, 2, 3, 5, 7, and 8. When the two dice are tossed, what is the probability that their sum will be an even number?

(A) $\frac{4}{9}$ (B) $\frac{1}{2}$ (C) $\frac{5}{9}$ (D) $\frac{3}{5}$ (E) $\frac{2}{3}$

(2) (AMC 8, 2019, Problem 6) There are 81 grid points (uniformly spaced) in the square shown in the diagram below, including the points on the edges. Point P is in the center of the square. Given that point Q is randomly chosen among the other 80 points, what is the probability that the line PQ is a line of symmetry for the square?

(A) $\frac{1}{5}$ (B) $\frac{1}{4}$ (C) $\frac{2}{5}$ (D) $\frac{9}{20}$ (E) $\frac{1}{2}$

(3) (AMC 8, 2018, Problem 23) From a regular octagon, a triangle is formed by connecting three randomly chosen vertices of the octagon. What is the probability that at least one of the sides of the triangle is also a side of the octagon?

(A) $\frac{2}{7}$ (B) $\frac{5}{42}$ (C) $\frac{11}{14}$ (D) $\frac{5}{7}$ (E) $\frac{6}{7}$

(4) (AMC 8, 2016, Problem 13) Two different numbers are randomly selected from the set $-2, -1, 0, 3, 4, 5$ and multiplied together. What is the probability that the product is 0?

(A) $\frac{1}{6}$ (B) $\frac{1}{5}$ (C) $\frac{1}{4}$ (D) $\frac{1}{3}$ (E) $\frac{1}{2}$

(5) (AMC 8, 2017, Problem 10) A box contains five cards, numbered $1, 2, 3, 4$ and 5. Three cards are selected randomly without replacement from the box. What is the probability that 4 is the largest value selected?

(A) $\frac{1}{10}$ (B) $\frac{1}{5}$ (C) $\frac{3}{10}$ (D) $\frac{2}{5}$ (E) $\frac{1}{2}$.

(6) (AMC 8, 2013, Problem 8) A fair coin is tossed 3 times. What is the probability of at least two consecutive heads?

(A) $\frac{1}{8}$ (B) $\frac{1}{4}$ (C) $\frac{3}{8}$ (D) $\frac{1}{2}$ (E) $\frac{3}{4}$.

(7) (AMC 8, 2018, Problem 11) Abby, Bridget, and four of their classmates will be seated in two rows of three for a group picture

$$\begin{array}{ccc} X & X & X \\ X & X & X \end{array}$$

If the seating positions are assigned randomly, what is the probability that Abby and Bridget are adjacent to each other in the same row or the same column?

(A) $\frac{1}{3}$ (B) $\frac{2}{5}$ (C) $\frac{7}{15}$ (D) $\frac{1}{2}$ (E) $\frac{2}{3}$

(8) (AMC 8, 2016, Problem 21) A top hat contains 3 red chips and 2 green chips. Chips are drawn randomly, one at a time without replacement until all 3 of the reds are drawn or until both green chips are drawn. What is the probability that the 3 reds are drawn?

(A) $\frac{3}{10}$ (B) $\frac{2}{5}$ (C) $\frac{1}{2}$ (D) $\frac{3}{5}$ (E) $\frac{7}{10}$

(9) (AMC 8, 2015, Problem 7) Each of two boxes contains three chips numbered 1, 2, 3. A chip is drawn randomly from each box and the numbers on the two chips are multiplied. What is the probability that their product is even?

(A) $\frac{1}{9}$ (B) $\frac{2}{9}$ (C) $\frac{4}{9}$ (D) $\frac{1}{2}$ (E) $\frac{5}{9}$

(10) (AMC 8, 2014, Problem 12) A magazine printed photos of three celebrities along with three photos of the celebrities as babies. The baby pictures did not identify the celebrities. Readers were asked to match each celebrity with the correct baby pictures. What is the probability that a reader guessing at random will match all three correctly?

(A) $\frac{1}{9}$ (B) $\frac{1}{6}$ (C) $\frac{1}{4}$ (D) $\frac{1}{3}$ (E) $\frac{1}{2}$

More problems.

(Conditional probability) Sam rolled a fair dice and tell me the number is even. What is the probability that the number is two? What is the probability that the number is three or four?

(A problem on https://math.stackexchange.com/questions/3263051/probability-dilemma) We flip a coin. If it was Heads, we roll a die and if it was Tails, we flip three other coins. What's the probability of exactly having one coin as Heads?

(Three prisoners problem) A, B, and C are in separate cells and sentenced to death. The governor has selected one of them at random to be pardoned. The warden knows which one is pardoned, but is not allowed to tell. Prisoner A begs the warden to let him know the identity of one of the two who are going to be executed. "If B is to be pardoned, give me C's name. If C is to be pardoned, give me B's name. And if I'm to be pardoned, secretly flip a coin to decide whether to name B or C." The warden tells A that B is to be executed. Prisoner A is pleased because he believes that his probability of surviving has gone up from 1/3 to 1/2 as it is now between him and C.

Prisoner A secretly tells C the news, who is also pleased, because he reasons that A still has a chance of 1/3 to be the pardoned one, but his chance has gone up to 2/3. Who is correct?