Cupertino Math Circle: Probability

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1 Introduction

What is a probability? A probability indicates the chance that an event will happen. A probability can be any number from 0 to 1. A probability of 0 means that an event will never happen. A probability of $\frac{1}{2}$ means that an event is equally likely to happen or not happen. A probability of 1 means that an event will certainly happen.

The probability that event A will happen is written as $\mathbb{P}(A)$. The probability that event A will not happen is $1 - \mathbb{P}(A)$. The fundamental formula for computing probability is

$$P(A) = \frac{\text{ways } A \text{ can happen}}{\text{possible outcomes}}.$$

Some notation in Combinatorics: The way of choosing m numbers out of $1, 2, \cdots, n$, (the order matters) is

$$n \times (n-1) \times \cdots (m+1) = \frac{n!}{m!}$$

The way of choosing m numbers out of $1, 2, \cdots, n$ (the order does not matters) is

$$\binom{n}{m} = \frac{n \times (n-1) \times \cdots \times (n-m+1)}{m!} = \frac{n!}{m!(n-m)!}$$

Some identities:

$$\binom{n}{m} = \binom{n}{n-m}, \qquad \binom{n}{m} = \binom{n}{m-1} + \binom{n-1}{m-1}.$$

2 Some exercises

(1) (AMC 10A 2018 Problem 11) When 7 fair standard 6-sided dice are thrown, the probability that the sum of the numbers on the top faces is 10 can be written as $\frac{n}{6^7}$, where n is a positive integer. What is n?

(A) 42 (B) 49 (C) 56 (D) 63 (E) 84

(2) (AMC 10B 2019 Problem 17) A red ball and a green ball are randomly and independently tossed into bins numbered with positive integers so that for each ball, the probability that it is tossed into bin k is 2^{-k} for $k = 1, 2, 3, \ldots$. What is the probability that the red ball is tossed into a higher-numbered bin than the green ball?

(A) $\frac{1}{4}$ (B) $\frac{2}{7}$ (C) $\frac{1}{3}$ (D) $\frac{3}{8}$ (E) $\frac{3}{7}$

(3) (2014 AIME I Problems/Problem 2) An urn contains 4 green balls and 6 blue balls. A second urn contains 16 green balls and N blue balls. A single ball is drawn at random from each urn. The probability that both balls are of the same color is 0.58. Find N.

(4) (AMC 10A 2020 Problem 13) A frog sitting at the point (1,2) begins a sequence of jumps, where each jump is parallel to one of the coordinate axes and has length 1, and the direction of each jump (up, down, right, or left) is chosen independently at random. The sequence ends when the frog reaches a side of the square with vertices (0,0),(0,4),(4,4) and (4,0). What is the probability that the sequence of jumps ends on a vertical side of the square?

(A) $\frac{1}{2}$ (B) $\frac{5}{8}$ (C) $\frac{2}{3}$ (D) $\frac{3}{4}$ (E) $\frac{7}{8}$

(5) (AMC 10A 2020 Problem 25) Jason rolls three fair standard six-sided dice. Then he looks at the rolls and chooses a subset of the dice (possibly empty, possibly all three dice) to reroll. After rerolling, he wins if and only if the sum of the numbers face up on the three dice is exactly 7. Jason always plays to optimize his chances of winning. What is the probability that he chooses to reroll exactly two of the dice?

(A) $\frac{7}{36}$ (B) $\frac{5}{24}$ (C) $\frac{2}{9}$ (D) $\frac{17}{72}$ (E) $\frac{1}{4}$

(6) (2017 AMC 12B Problems/Problem 20)

Real numbers x and y are chosen independently and uniformly at random from the interval $(0,1)$. What is the probability that $\lfloor \log_2 x \rfloor = \lfloor \log_2 y \rfloor$?

(A) $\frac{1}{8}$ (B) $\frac{1}{6}$ (C) $\frac{1}{4}$ (D) $\frac{1}{3}$ (E) $\frac{1}{2}$

(7) (2019 AIME I Problems/Problem 5) A moving particle starts at the point $(4,4)$ and moves until it hits one of the coordinate axes for the first time. When the particle is at the point (a,b), it moves at random to one of the points $(a-1,b)$, $(a,b-1)$, or $(a-1,b-1)$, each with probability $\frac{1}{3}$, independently of its previous moves. The probability that it will hit the coordinate axes at $(0,0)$ is $\frac{m}{3^n}$, where m and n are positive integers such that m is not divisible by 3. Find $m+n$.

(8) (2018 AIME II Problems/Problem 13) Misha rolls a standard, fair six-sided die until she rolls $1,2,3$ in that order on three consecutive rolls. The probability that she will roll the die an odd number of times is $\dfrac{m}{n}$ where m and n are relatively prime positive integers. Find $m+n$.

A more challenging problem.

Inclusion–exclusion principle: For two events A, B, let $A \cup B$ denote the event that at least A or B happens, let $A \cap B$ denote the event that A, B both happen. We have

$$\mathbb{P}\Big(\bigcup_{i=1}^{n} A_i\Big) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} \mathbb{P}\Big(\bigcap_{j=1}^{k} A_{i_j}\Big)$$

Ten bald men wearing hats go to a party, and when they arrived they put their hats in a dark closet. During the party, someone yells, "Fire!" and the men rush to the closet, grab a hat at random, and leave. What is the probability that

(1) Exactly nine gets the right hats?

(2) All get the wrong hat?