Some recent progress in the weak noise theory of the KPZ equation

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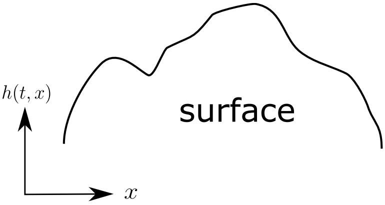
Joint work with Pierre Yves Gaudreau Lamarre and Li-Cheng Tsai

The KPZ equation

- The KPZ equation is a stochastic PDE that models the random surface growth:

$$\partial_t h = \frac{1}{2}\partial_{xx} h + \frac{1}{2}(\partial_x h)^2 + \xi.$$

The growth subjects to smoothing mechanism, slope dependence, and space-time independent noise.



Long time regime

[Amir-Corwin-Quastel 11] Under narrow wedge initial data,

$$\frac{h(2t,0)+\frac{t}{12}}{t^{1/3}} \stackrel{t\to\infty}{\to} \text{Tracy-Widom GUE.}$$

Other results for the KPZ equation such as tail bounds, one-point large deviations, law of iterated the logarithm ... have been studied.

We study the KPZ equation in a different scenario.

The theory of Freidlin-Wentzell studies the large deviation principle (LDP) of small noise perturbation of dynamical systems.

In our context, we consider the KPZ equation with a small parameter ε,

$$\partial_t h_\varepsilon = \tfrac{1}{2}\partial_{xx} h_\varepsilon + \tfrac{1}{2}(\partial_x h_\varepsilon)^2 + \sqrt{\varepsilon}\xi.$$

The formalism is standard in stochastic analysis literature.

Short time formalism

The process $h_\varepsilon(t,x)$ has the same law as $h_1(\varepsilon^2 t, \varepsilon x)$.

The Freidlin-Wentzell LDP

The Freidlin-Wentzell LDP studies the atypical behavior of h_ε.

The typical behavior

As $\varepsilon \to 0$, $h_\varepsilon \to \mathsf{h}_0$, where h_0 solves the HJ equation

$$\partial_t \mathsf{h}_0 = \tfrac{1}{2}\partial_{xx}\mathsf{h}_0 + \tfrac{1}{2}(\partial_x \mathsf{h}_0)^2.$$

The atypical behavior (one-point)

Fix $(t, x) = (2, 0)$. Study the probability $\mathbb{P}\big(h_\varepsilon(2,0) \approx \lambda\big)$, where $\lambda \neq \mathsf{h}_0(2,0)$.

The one-point LDP

The one-point LDP refers to, as $\varepsilon \to 0$,

$$\mathbb{P}\big(h_\varepsilon(2,0) \approx \lambda\big) \approx \exp\Big(-\varepsilon^{-1} I_{op}(\lambda)\Big),$$

where I_{op} is referred as the (one-point) rate function.

Precise meaning

More precisely, the above notation means

$$\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \varepsilon \log \mathbb{P}\big(|h_\varepsilon(2,0) - \lambda| \le \delta\big) = -I_{op}(\lambda).$$

The Freidlin-Wentzell LDP

The Freidlin-Wentzell (process-level) LDP: For $g \in C([0,2] \times \mathbb{R})$,

$$\mathbb{P}\big(h_\varepsilon \approx g\big) \approx e^{-\varepsilon^{-1} I(g)}.$$

Precise meaning

For any measurable set $\Omega \subseteq C([0,2] \times \mathbb{R})$, we have

$$-\inf_{g \in \Omega^\circ} I(g) \leq \liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P}\big(h_\varepsilon \in \Omega\big) \leq \limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}\big(h_\varepsilon \in \Omega\big) \leq -\inf_{g \in \overline{\Omega}} I(g).$$

LDP and most probable shape

Condition on $h_\varepsilon(2,0)=\lambda$, we want to study the $\varepsilon \to 0$ limit of h_ε, which is called the most probable shape.

The most probable shape is given by

$$\arg\min \Big\{ I(g) : g(2,0) = \lambda \Big\}.$$

Our main result is the $\lambda \to \pm\infty$ limit of the most probable shape.

Different approaches

Exact formula approach

Use the exact formula to study the one-point LDP rate function.

Obtain the exact one-point rate function.

Krajenbrink, Le Doussal, Majumdar, Rosso, Schehr, ...

Weak noise theory approach

Study the process-level LDP and most probable shapes.

Hartmann, Janas, Kamenev, Katzav, Kolokolov, Kurshnov, Meerson, Sasorov, ...

Gaudreau Lamarre, L, Tsai, ...

Theorem (L-Tsai 21)

Take $g \in C([0,2] \times \mathbb{R})$, *we have*

$$\mathbb{P}\big(h_\varepsilon \approx g\big) \approx e^{-\varepsilon^{-1} I(g)},$$

the rate function I *will be specified later.*

The rate function

The definition of I

It is known that for $\rho \in L^2([0,2] \times \mathbb{R})$,

$$\mathbb{P}\big(\sqrt{\varepsilon}\xi \approx \rho\big) \approx e^{-\varepsilon^{-1} \cdot \frac{1}{2}\|\rho\|_{L^2}^2}.$$

We define $\mathsf{H}[\rho]$, the solution to

$$\partial_t \mathsf{H} = \tfrac{1}{2}\partial_{xx}\mathsf{H} + \tfrac{1}{2}(\partial_x \mathsf{H})^2 + \rho.$$

The rate function is given by

$$I(g) = \inf\Big\{\tfrac{1}{2}\|\rho\|_{L^2}^2 : \mathsf{H}[\rho] = g\Big\}.$$

The minimizer of variational formula

By the expression of I, we have

$$\inf\left\{I(g):g(2,0)=\lambda\right\}$$
$$=\inf\left\{\frac{1}{2}\|\rho\|_{L^2}^2:\mathsf{H}[\rho](2,0)=\lambda\right\}.$$

Proposition (Gaudreau Lamarre-L-Tsai 21)

The minimizer of the inf exists.

The minimizer of variational formula

$$\inf \left\{ \tfrac{1}{2} \|\rho\|_{L^2}^2 : \mathsf{H}[\rho](2,0) = \lambda \right\}.$$

Question (Uniqueness?)

It depends on the initial condition and value of λ.

When λ is small, the uniqueness holds.

When λ is large, whether we have uniqueness depends on the initial data. For the Brownian initial condition, a symmetry breaking has been predicted by Hartmann, Janas, Kamenev, Krajenbrink, Le Doussal, Meerson, Sasorov ...

Narrow wedge initial data

The Hopf-Cole solution

The solution to the KPZ equation

$$\partial_t h = \tfrac{1}{2}\partial_{xx} h + \tfrac{1}{2}(\partial_x h)^2 + \xi$$

can be defined via $h = \log Z$, where Z solves the stochastic heat equation

$$\partial_t Z = \tfrac{1}{2}\partial_{xx} Z + \xi Z.$$

Narrow wedge initial data

We set $Z(0,\cdot) = \delta(\cdot)$. As $t \to 0$,

$$h(t,x) \approx -\tfrac{x^2}{2t} - \log\sqrt{2\pi t}.$$

Feynman-Kac formula

The solution to the SHE with Dirac-delta initial condition admits a Feynman-Kac formula [Amir-Corwin-Quastel 11],

$$Z(t,x) = \mathbb{E}\Big[:\exp: \Big(\int_0^t \xi(s, W_x(s)) ds \Big) p(t,x) \Big],$$

where W_x is a Brownian bridge with $W_x(0) = 0$, $W_x(t) = x$ and $p(t,x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$.

The Wick exponential can be interpreted as chaos expansion

$$Z(t,x) = \sum_{n=0}^{\infty} \int_{\Delta_n(t)} \int_{\mathbb{R}^{n+1}} p(s_n - s_{n+1}, y_n) \prod_{i=1}^{n} p(s_{i-1} - s_i, y_{i-1} - y_i) \xi(s_i, y_i) ds_i dy_i.$$

where $\Delta_n(t) = \{0 = s_{n+1} < \cdots < s_1 < s_0 = t\}$ and $y_0 = x$.

Most probable shape

Define the minimizer of

$$\inf \left\{ \tfrac{1}{2}\|\rho\|_{L^2}^2 : \mathsf{H}[\rho](2,0) = \lambda \right\}$$

as ρ_λ.

Let $(w_\lambda, \mathsf{h}_\lambda)$ be a scaled version of $(\rho_\lambda, \mathsf{H}[\rho_\lambda])$,

$$w_\lambda(t,x) := |\lambda|^{-1}\rho_\lambda(\cdot, |\lambda|^{1/2}\cdot),$$
$$\mathsf{h}_\lambda(t,x) := |\lambda|^{-1}\mathsf{H}[\rho_\lambda](\cdot, |\lambda|^{1/2}\cdot).$$

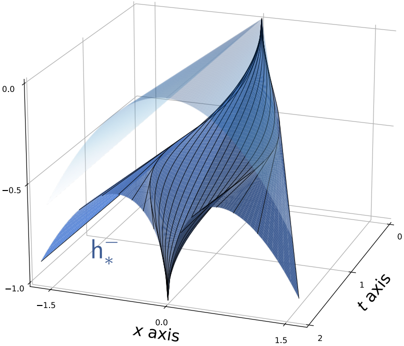
The lower-tail ($\lambda \to -\infty$) limit

Theorem (L-Tsai 22)

Uniformly over the compact domain in $[0,2] \times \mathbb{R}$, *we have*

$$\lim_{\lambda \to -\infty} \mathsf{h}_\lambda = \mathsf{h}_*^-,$$

where h_*^- *is explicit and will be specified later.*



Courtsey to Li-Cheng Tsai

The weak noise theory approach

Recall that $(w_\lambda, \mathsf{h}_\lambda)$ is a scaled version of $(\rho_\lambda, \mathsf{H}[\rho_\lambda])$. The $(w_\lambda, \mathsf{h}_\lambda)$ satisfies a pair of PDEs

$$\partial_t \mathsf{h}_\lambda - (2|\lambda|)^{-1} \partial_{xx} \mathsf{h}_\lambda - \frac{1}{2}(\partial_x \mathsf{h}_\lambda)^2 = w_\lambda,$$
$$\partial_t w_\lambda + (2|\lambda|)^{-1} \partial_{xx} w_\lambda = \partial_x (w_\lambda \partial_x \mathsf{h}_\lambda).$$

[Kamenev-Meerson-Sasorov 16]: Taking $\lambda \to -\infty$ limit and solving the limiting equation, one gets

$$w_*(t,x) = -\tfrac{1}{2\pi \ell(t)} \Big(1 - \tfrac{x^2}{\ell(t)^2}\Big)_+,$$

where ℓ is defined implicitly via

$$|t-1| = \sqrt{1 - \tfrac{\pi \ell(t)}{2}} + \tfrac{2}{\pi} \tan^{-1} \sqrt{\tfrac{2}{\pi \ell(t)} - 1}.$$

An L^2-convergence

Proposition (L-Tsai 21)

As $\lambda \to -\infty$, $w_\lambda \to w_*$ *in* L^2.

The lower-tail ($\lambda \to -\infty$) limit

By Feynman-Kac formula,

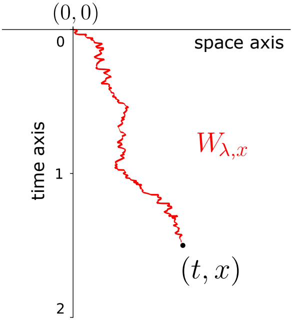
$$\mathsf{h}_\lambda(t,x) = |\lambda|^{-1} \log \mathbb{E}\Big[\exp\Big(|\lambda| \int_0^t w_\lambda(s, W_{\lambda,x}(s)) ds \Big) p_\lambda(t,x) \Big],$$

where $W_{\lambda,x}$ is a Brownian bridge with diffusive parameter $|\lambda|^{-1/2}$ and $p_\lambda(t,x) = \frac{1}{\sqrt{2\pi|\lambda|^{-1}t}} e^{-\frac{|\lambda|x^2}{2t}}$.

Replacing w_λ with w_* and taking $\lambda \to -\infty$, we have

$$\mathsf{h}_*^-(t,x) = \sup_\gamma \Big(\int_0^t \underbrace{w_*(s,\gamma(s))}_{\text{energy}} - \underbrace{\tfrac{1}{2}\dot\gamma(s)^2}_{\text{entropy}} ds \Big).$$

Maximize the combination of energy and entropy.

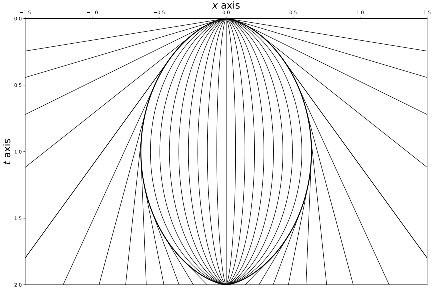


A class of geodesics

We call the maximizers as **geodesics**.

Proposition (L-Tsai 21)

For any $(t, x) \in (0, 2] \times \mathbb{R}$, the geodesics from (t, x) to $(0, 0)$ are as depicted in the figure.



Some remarks on the proof

- Knowing the L^2-convergence $w_\lambda \to w_*$ is not enough. One can find $\widetilde{w}_\lambda \to w_*$ in L^2 and $\mathsf{h}_\lambda(\widetilde{w}_\lambda) \nrightarrow \mathsf{h}_*^-$.

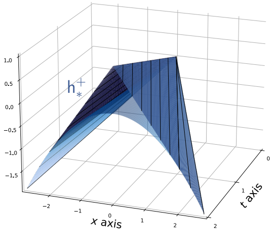
- The L^2-convergence is enough for proving a lower bound $\liminf_{\lambda\to-\infty} \mathsf{h}_\lambda(\widetilde{w}_\lambda) \geq \mathsf{h}_*^-$.

- The upper bound $\limsup_{\lambda\to-\infty} \mathsf{h}_\lambda(w_\lambda) \leq \mathsf{h}_*^-$ requires the property of $\mathsf{h}_\lambda(w_\lambda)(2,0) = -1$.

The upper-tail ($\lambda \to +\infty$) limit.

In [Gaudreau Lamarre-L-Tsai 21], we also obtain h_*^+: The $\lambda \to +\infty$ limit of most probable shape.

- Show that a scaled version w_λ is nearly time-independent and approximates sech^2.
- Study the ground state of Schrödinger-type operator in the variational formula.
- Identify the ground state using Gagliardo-Nirenberg-Sobolev inequality.



The $\lambda \to \pm\infty$ limit

- For $\lambda \to -\infty$, w_λ spans a wide region in spacetime.
- For $\lambda \to \infty$, w_λ vanishes everywhere except in a small neighborhood around $x = 0$.

Minimize $\|w_\lambda\|_{L^2}$ so that

$$\mathbb{E}\left[\exp\left(|\lambda| \int_0^2 w_\lambda(s, W_\lambda(s))ds\right)\right] = e^\lambda,$$

where W_λ is a Brownian bridge with diffusive parameter $|\lambda|^{-1/2}$

Some open problems

- Generalize the theory for other initial data, for example, Brownian initial data.
- The uniqueness of the minimizer and symmetry breaking.
- Apply this approach to other Stochastic PDEs.

Thank you!