

Some recent progress in the weak noise theory of the KPZ equation

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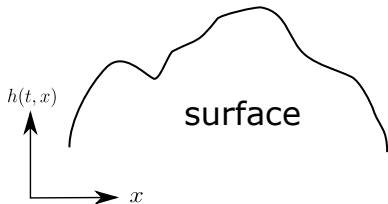
Joint work with Pierre Yves Gaudreau Lamarre and Li-Cheng Tsai

The KPZ equation

- The KPZ equation is a stochastic PDE that models the random surface growth:

$$\partial_t h = \frac{1}{2} \partial_{xx} h + \frac{1}{2} (\partial_x h)^2 + \xi.$$

The growth subjects to **smoothing mechanism**, **slope dependence**, and **space-time independent noise**.



Long time regime: Results

There has been a lot of progress in understanding the behavior of $h(t, \cdot)$ as $t \rightarrow \infty$.

Fluctuation limit theorem: [Amir-Corwin-Quastel 11], [Quastel-Sarkar 23], [Virag 20], ...

Tail behaviors: [Corwin-Ghosal 20a, 20b], [Tsai 22], [Das-Tsai 21], [Cafasso-Claeys 22], [Ganguly-Hegde 22], ...

Law of iterated logarithm: [Das-Ghosal 21], ...

Polymer Localization: [Das-Zhu 22], ...

Freidlin-Wentzell Large Deviation Principle (LDP)

The theory of Freidlin-Wentzell studies the **large deviation principle (LDP)** of **small noise perturbation** of dynamical systems.

In our context, we consider the KPZ equation with a **small parameter** ε ,

$$\partial_t h_\varepsilon = \frac{1}{2} \partial_{xx} h_\varepsilon + \frac{1}{2} (\partial_x h_\varepsilon)^2 + \sqrt{\varepsilon} \xi.$$

The formalism is standard in stochastic analysis literature.

Short time formalism

The process $h_\varepsilon(t, x)$ has the same law as $h_1(\varepsilon^2 t, \varepsilon x)$.

The Freidlin-Wentzell LDP

The Freidlin-Wentzell LDP studies the **atypical behavior** of h_ε .

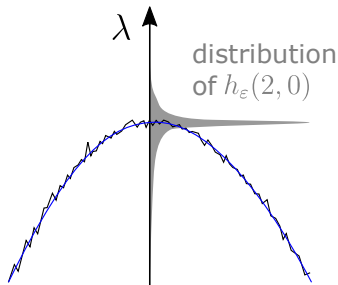
The typical behavior

As $\varepsilon \rightarrow 0$, $h_\varepsilon \rightarrow h_0$, where h_0 solves the HJ equation

$$\partial_t h_0 = \frac{1}{2} \partial_{xx} h_0 + \frac{1}{2} (\partial_x h_0)^2.$$

The atypical behavior (one-point)

Fix $(t, x) = (2, 0)$. Study the probability $\mathbb{P}(h_\varepsilon(2, 0) \approx \lambda)$, where $\lambda \neq h_0(2, 0)$.



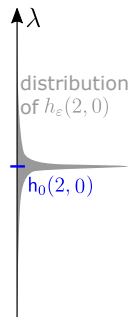
courtesy of Li-Cheng

The one-point LDP

The one-point LDP refers to, as $\varepsilon \rightarrow 0$,

$$\mathbb{P}(h_\varepsilon(2,0) \approx \lambda) \approx \exp\left(-\varepsilon^{-1} I_{op}(\lambda)\right),$$

where I_{op} is referred as the (one-point) **rate function**.



Precise meaning

More precisely, the above notation means

$$\lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}(|h_\varepsilon(2,0) - \lambda| \leq \delta) = -I_{op}(\lambda).$$

The Freidlin-Wentzell LDP

The Freidlin-Wentzell (process-level) LDP: For $g \in C([0, 2] \times \mathbb{R})$,

$$\mathbb{P}(h_\varepsilon \approx g) \approx e^{-\varepsilon^{-1}I(g)}.$$

Precise meaning

For any measurable set $\Omega \subseteq C([0, 2], \mathbb{R})$, we have

$$-\inf_{g \in \Omega^\circ} I(g) \leq \liminf_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}(h_\varepsilon \in \Omega) \leq \limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}(h_\varepsilon \in \Omega) \leq -\inf_{g \in \bar{\Omega}} I(g).$$

LDP and most probable shape

Condition on $h_\varepsilon(2, 0) = \lambda$, we want to study the $\varepsilon \rightarrow 0$ limit of h_ε , which is called **the most probable shape**.

The most probable shape is given by

$$\arg \min \left\{ I(g) : g(2, 0) = \lambda \right\}.$$

Our main results are the $\lambda \rightarrow \pm\infty$ asymptotic of the most probable shape.

Different approaches

Exact formula approach

Use the exact formula to study the one-point LDP rate function.

Obtain the exact one-point rate function.

Krajenbrink, Le Doussal, Majumdar, Rosso, Schehr, ...

Weak noise theory approach

Study the process-level LDP and most probable shapes.

Hartmann, Janas, Kamenev, Katzav, Kolokolov, Kurshnov, Meerson, Sasorov, ...

Gaudreau Lamarre, L, Tsai, ...

The weak noise theory + integrable system (2021+)

Bettelheim, Krajenbrink, Le Doussal, Meerson, Smith, ...

Tsai, ...

Theorem (L-Tsai 21)

Take $g \in C([0, 2] \times \mathbb{R})$, we have

$$\mathbb{P}(h_\varepsilon \approx g) \approx e^{-\varepsilon^{-1}I(g)},$$

the rate function I will be specified later.

The rate function

The definition of I

It is known that for $\rho \in L^2([0, 2] \times \mathbb{R})$,

$$\mathbb{P}(\sqrt{\varepsilon}\xi \approx \rho) \approx e^{-\varepsilon^{-1} \cdot \frac{1}{2} \|\rho\|_{L^2}^2}.$$

We define $\mathbf{H}[\rho]$, the solution to

$$\partial_t \mathbf{H} = \frac{1}{2} \partial_{xx} \mathbf{H} + \frac{1}{2} (\partial_x \mathbf{H})^2 + \rho.$$

The rate function is given by

$$I(g) = \inf \left\{ \frac{1}{2} \|\rho\|_{L^2}^2 : \mathbf{H}[\rho] = g \right\}.$$

The minimizer of variational formula

By the expression of I , we have

$$\begin{aligned} & \inf \{ I(g) : g(2, 0) = \lambda \} \\ &= \inf \left\{ \frac{1}{2} \|\rho\|_{L^2}^2 : \mathbf{H}(\rho)(2, 0) = \lambda \right\}. \end{aligned}$$

Proposition (Gaudreau Lamarre-L-Tsai 23)

The minimizer of the inf exists.

The minimizer of variational formula

$$\inf \left\{ \frac{1}{2} \|\rho\|_{L^2}^2 : \mathbf{H}(\rho)(2, 0) = \lambda \right\}.$$

Question (Uniqueness?)

It depends on the initial condition and value of λ .

When λ is small, the uniqueness holds.

*When λ is large, whether we have uniqueness depends on the initial data. For the Brownian initial condition, a symmetry breaking has been predicted by *Hartmann, Janas, Kamenev, Krajenbrink, Le Doussal, Meerson, Sasorov ...**

Narrow wedge initial data

The Hopf-Cole solution

The solution to the KPZ equation

$$\partial_t h = \frac{1}{2} \partial_{xx} h + \frac{1}{2} (\partial_x h)^2 + \xi$$

can be defined via $h = \log Z$, where Z solves the stochastic heat equation

$$\partial_t Z = \frac{1}{2} \partial_{xx} Z + \xi Z.$$

Narrow wedge initial data

We set $Z(0, \cdot) = \delta(\cdot)$. As $t \rightarrow 0$,

$$h(t, x) \approx -\frac{x^2}{2t} - \log \sqrt{2\pi t}.$$

Most probable shape

Define the minimizer of

$$\inf \left\{ \frac{1}{2} \|\rho\|_{L^2}^2 : \mathbf{H}(\rho)(2, 0) = \lambda \right\}$$

as ρ_λ .

Let $(w_\lambda, \mathbf{h}_\lambda)$ be a scaled version of $(\rho_\lambda, \mathbf{H}(\rho_\lambda))$,

$$w_\lambda(t, x) := |\lambda|^{-1} \rho_\lambda(\cdot, |\lambda|^{1/2} \cdot),$$

$$\mathbf{h}_\lambda(t, x) := |\lambda|^{-1} \mathbf{H}(\rho_\lambda)(\cdot, |\lambda|^{1/2} \cdot).$$

When $\lambda < 0$, $\mathbf{h}_\lambda(2, 0) = -1$.

A lower-tail ($\lambda \rightarrow -\infty$) limit

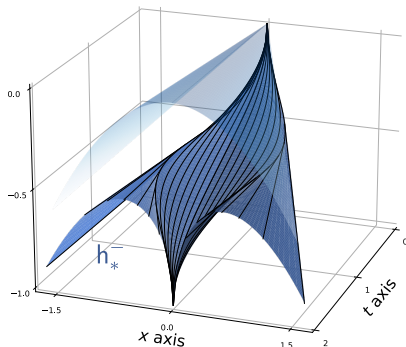
Theorem (L-Tsai 22)

Uniformly over the compact domain in $[0, 2] \times \mathbb{R}$, we have

$$\lim_{\lambda \rightarrow -\infty} h_\lambda = h_*^-,$$

where h_*^- is explicit and will be specified later.

This confirms the prediction from Kamenev, Kolokolov, Korshunov, Meerson, Sasorov, ...



A lower-tail ($\lambda \rightarrow -\infty$) limit

The (w_λ, h_λ) satisfies a pair of PDEs

$$\begin{aligned}\partial_t h_\lambda - (2|\lambda|)^{-1} \partial_{xx} h_\lambda - \frac{1}{2} (\partial_x h_\lambda)^2 &= w_\lambda, \\ \partial_t w_\lambda + (2|\lambda|)^{-1} \partial_{xx} w_\lambda &= \partial_x (w_\lambda \partial_x h_\lambda).\end{aligned}$$

[Kamenev-Meerson-Sasorov 16]: Taking $\lambda \rightarrow -\infty$ limit and solving the limiting equation, one gets

$$w_*(t, x) = -\frac{1}{2\pi\ell(t)} \left(1 - \frac{x^2}{\ell(t)^2}\right)_+,$$

where ℓ solves the equation

$$|t - 1| = \sqrt{1 - \frac{\pi\ell(t)}{2}} + \frac{2}{\pi} \tan^{-1} \sqrt{\frac{2}{\pi\ell(t)} - 1}.$$

Proposition (L-Tsai 21)

As $\lambda \rightarrow -\infty$, $w_\lambda \rightarrow w_*$ in L^2 .

The lower-tail ($\lambda \rightarrow -\infty$) limit

By Feynman-Kac formula,

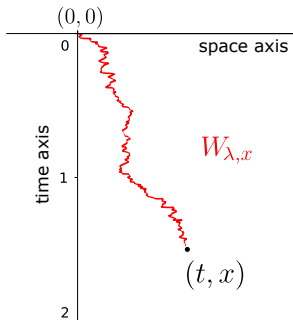
$$h_\lambda(t, x) = |\lambda|^{-1} \log \mathbb{E} \left[\exp \left(|\lambda| \int_0^t w_\lambda(s, W_{\lambda, x}(s)) ds \right) p_\lambda(t, x) \right],$$

where $W_{\lambda, x}(s)$ is a Brownian bridge with diffusive parameter $|\lambda|^{-1/2}$ and $p_\lambda(t, x) = \frac{1}{\sqrt{2\pi|\lambda|^{-1}}} e^{-\frac{|\lambda|x^2}{2t}}$.

Replacing w_λ with w_* and taking $\lambda \rightarrow -\infty$, we have

$$h_*^-(t, x) = \sup_\gamma \left(\int_0^t \underbrace{w_*(s, \gamma(s))}_{\text{energy}} - \underbrace{\frac{1}{2} \dot{\gamma}(s)^2}_{\text{entropy}} ds \right).$$

Maximize the combination of energy and entropy.

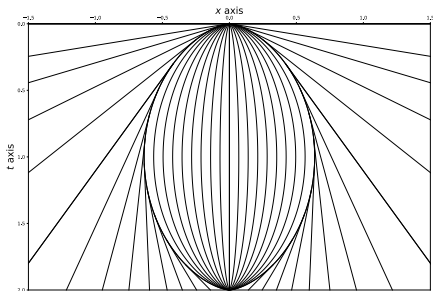


A class of geodesics

We call the maximizers as **geodesics**.

Proposition (L-Tsai 21)

For any $(t, x) \in (0, 2] \times \mathbb{R}$, the geodesics from (t, x) to $(0, 0)$ are as depicted in the figure.



Some remarks on the proof

Knowing the L^2 convergence $w_\lambda \rightarrow w_*$ is not enough. One can find $\tilde{w}_\lambda \rightarrow w_*$ in L^2 and $h_\lambda(\tilde{w}_\lambda) \not\rightarrow h_*^-$.

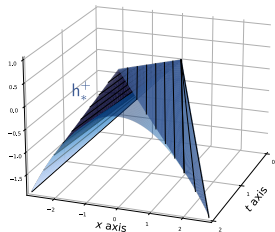
Sketch: The L^2 convergence is enough for proving a lower bound $\liminf_{\lambda \rightarrow -\infty} h_\lambda(\tilde{w}_\lambda) \geq h_*^-$.

One step to prove the upper bound requires the equi-continuity of $h_\lambda(\tilde{w}_\lambda)$.

The $\lambda \rightarrow +\infty$ limit.

In [Gaudreau Lamarre-L-Tsai 23], we also obtain h_*^+ : The $\lambda \rightarrow +\infty$ limit of most probable shape.

- Show that w_λ is nearly time-independent.
- Study the ground state of Schrödinger-type operator.
- Identify the ground state using Gagliardo-Nirenberg-Sobolev inequality.



Some more remarks

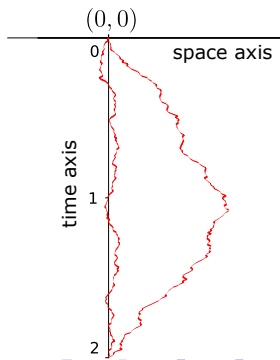
The $\lambda \rightarrow \pm\infty$ limit

- For $\lambda \rightarrow -\infty$, w_λ spans a wide region in spacetime.
- For $\lambda \rightarrow \infty$, w_λ vanishes everywhere except in a small neighborhood around $x = 0$.

Minimize $\|w_\lambda\|_{L^2}$ so that

$$\mathbb{E} \left[\exp \left(|\lambda| \int_0^2 w_\lambda(s, W_\lambda(s)) ds \right) \right] = e^\lambda,$$

where $W_\lambda(s) = |\lambda|^{-1/2} B_b(s)$.



Some open problems

Generalize the theory for other initial data, for example, Brownian initial data.

The uniqueness of the minimizer and symmetry breaking.

Extract the $\lambda \rightarrow -\infty$ limit using the method of integrable PDE.



Thank you!