

# Introduction to large deviation theory

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Oct 31, 2022

Oregon State University, Analysis Seminar

- Large deviation theory is one of the pillars of probability theory.
- Roughly speaking, large deviation theory concerns with the exponential decline of the probability measures of certain kinds of extreme or tail events.

# What are large deviations?

- We start with an example about tossing fair coins.
- We set a Rademacher random variable  $X = 1$  if we get a head, we set  $X = -1$  if we get a tail.

$$P(X = -1) = \frac{1}{2}, \quad P(X = 1) = \frac{1}{2}.$$

- We toss  $n$  fair coins and record the results as  $X_1, \dots, X_n$ . They are i.i.d. Rademacher random variables.
- **Law of large numbers:** Set  $S_n = X_1 + \dots + X_n$ , then

$$\lim_{n \rightarrow \infty} S_n/n = 0, \quad \text{a.s.}$$

- **Central limit theorem:** Study the fluctuation around 0 in the order of  $\frac{1}{\sqrt{n}}$ .

- The large deviation theory studies the behavior of  $\mathbb{P}\left(\left\{\frac{S_n}{n} \approx x\right\}\right)$  for fixed  $x$ , as  $n \rightarrow \infty$ .
- As special cases,

$$\mathbb{P}\left(\frac{S_n}{n} = -1\right) = 2^{-n} = e^{-n \log 2},$$

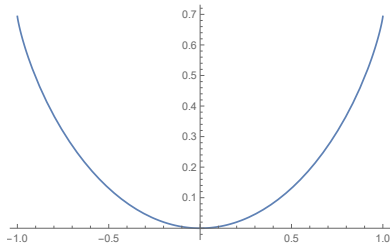
$$\mathbb{P}\left(\frac{S_n}{n} = 1\right) = 2^{-n} = e^{-n \log 2}.$$

- For general  $x$ ,

$$\mathbb{P}\left(\frac{S_n}{n} \approx x\right) \approx e^{-nI_{\text{Rad}}(x)},$$

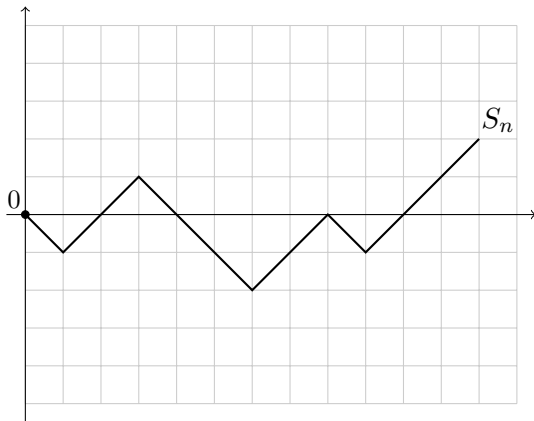
where **the rate function**

$$I_{\text{Rad}}(x) = \begin{cases} \log 2 + \frac{x+1}{2} \log \frac{x+1}{2} + \frac{1-x}{2} \log \frac{1-x}{2} & \text{if } x \in [-1, 1], \\ \infty & \text{else.} \end{cases}$$



# Simple random walk

- Let  $X_1, \dots, X_n, \dots$  be i.i.d. Rademacher random variables. Define the running sum  $S_k = \sum_{i=1}^k X_i$ . Note that we have studied the LDP of  $S_n$  in the coin flipping example.



# Simple random walk

- By law of large numbers, as  $n \rightarrow \infty$ ,

$$\left\{ \frac{S_{nt}}{n}, t \in [0, 1] \right\} \rightarrow 0.$$

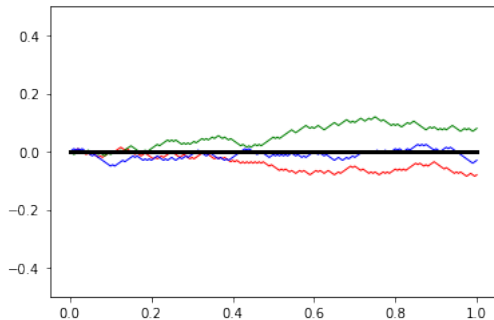


Figure 1: Samplings of simple random walk when  $n = 200$ .



- For nice  $\gamma \in C([0, 1])$  with  $\gamma(0) = 0$ , we have

$$\mathbb{P}\left(\left\{\frac{S_{nt}}{n}, t \in [0, 1]\right\} \approx \gamma\right) \approx e^{-nI_{\text{SRW}}(\gamma)},$$

where

$$I_{\text{SRW}}(\gamma) = \int_0^1 I_{\text{Rad}}(\dot{\gamma}(s)) ds.$$

# Most probable shape

- Conditioned on  $\frac{S_n}{n} = x$ , find **most probable shape** of  $\{\frac{S_{nt}}{n}, t \in [0, 1]\}$ , when  $n \rightarrow \infty$ ?

- The most probable shape refers to the path  $\gamma_*$  where

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(\left\|\frac{S_{n\cdot}}{n} - \gamma_*\right\| \leq \varepsilon \mid \frac{S_n}{n} = x\right) = 1, \quad \forall \varepsilon > 0.$$

- Intuitively, the most probable shape is  $\{tx, t \in [0, 1]\}$ .

# Most probable shape

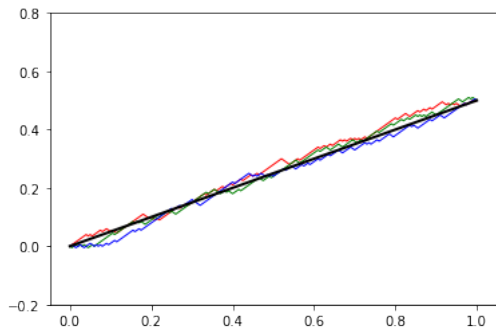


Figure 2: Samplings of simple random walk conditioned on  $S_n = 0.5n$  when  $n = 200$ .

- How to identify the most probable shape  $\gamma_*$  from the LDP?

$$\gamma_* = \arg \inf \left\{ I_{\text{SRW}}(\gamma) : \gamma(0) = 0, \gamma(1) = x \right\}.$$

Note that

$$I_{\text{SRW}}(\gamma) = \int_0^1 I_{\text{Rad}}(\dot{\gamma}(s)) ds.$$

By convexity of  $I_{\text{Rad}}$ , the minimizer  $\gamma_*$  should be a straight line,  $\gamma_*(t) = tx$ .

- Define  $B_t^k := \frac{S_{kt}}{\sqrt{k}}$ . As  $k \rightarrow \infty$ , the process  $B_t^k$  converges to a continuous process  $B_t$ , which is a **Brownian motion**.

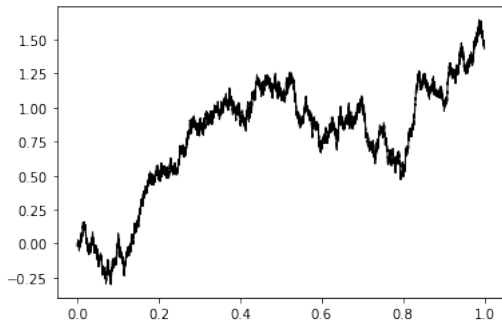


Figure 3: Simulation of  $B_t^k$ , when  $k = 50000$ .

The Brownian motion models the motion of the pollen particles as being moved by individual water molecules. It has the following properties:

- $B(0) = 0$ .
- $B$  has both stationary and independent increments.
- $B_t - B_s$  has a normal distribution with mean 0 and variance  $|t - s|$ .
- The process  $B_{nt}$  has the same law as  $\sqrt{n}B_t$ .

- Consider the Brownian motion starting from  $B_0 = 0$ . We study the LDP of  $\frac{B_n}{n}$ .
- The process has the same law as  $\sqrt{\varepsilon}B.$ ,  $\varepsilon = \frac{1}{n}$ . We have

$$\mathbb{P}\left(\sqrt{\varepsilon}B. \approx \gamma\right) \approx \exp\left(-\varepsilon^{-1}I_{\text{BM}}(\gamma)\right),$$

where

$$I_{\text{BM}}(\gamma) = \begin{cases} \int_0^1 \frac{1}{2} \dot{\gamma}(s)^2 ds, & \gamma(0) = 0 \text{ and } \gamma \in H^1([0, 1]), \\ \infty & \text{else.} \end{cases}$$

# Most probable shape

- Given  $\sqrt{\varepsilon}B_1 = y$ , what is the most probable shape?



# Finding the minimizer

- The most probable shape should be the minimizer of

$$\inf \left\{ I_{\text{BM}}(\gamma) : \gamma(0) = 0, \gamma(1) = y, \gamma \in H^1([0, 1]) \right\}$$

- We need to minimize

$$\int_0^1 \dot{\gamma}(s)^2 ds \text{ given } \gamma(0) = 0 \text{ and } \gamma(1) = y.$$

- The minimizer has constant derivative:  $\gamma(t) = ty$ .

# Spatial white noise

- Consider a Brownian motion  $B$ . The white noise  $\xi$  is the spatial derivative of it, i.e.  $\xi(t) = \frac{dB_t}{dt}$ .

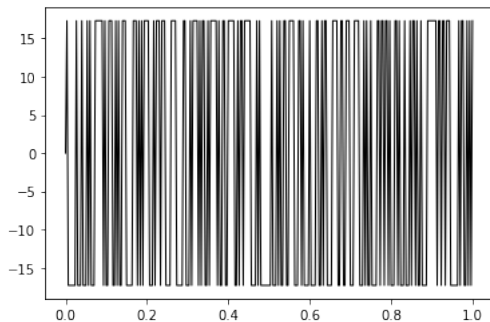


Figure 4: Simulation of spatial white noise

- Because  $B_t$  is not differentiable,  $\xi$  is a **generalized function**. For any compactly supported  $g$ ,

$$\int g\xi dt = - \int g'(t)B_t dt.$$

- Informally,  $\xi$  is a Gaussian process with correlation

$$\mathbb{E}[\xi(t)\xi(s)] = \delta(t - s).$$

- For  $\rho \in L^2(\mathbb{R})$ , as  $\varepsilon \rightarrow 0$ ,

$$\mathbb{P}(\sqrt{\varepsilon}\xi \approx \rho) \approx \exp\left(-\frac{1}{2}\varepsilon^{-1} \int_{\mathbb{R}} \rho^2\right).$$

- Informally, the space-time white noise is a two-dimensional Gaussian process  $\xi$  with correlation function

$$\mathbb{E}[\xi(t, x)\xi(s, y)] = \delta(t - s)\delta(x - y).$$

It can also be defined as the space-time derivative of a **Brownian sheet**.

- The LDP of space-time white noise, for  $\rho \in L^2(\mathbb{R}^2)$ ,

$$\mathbb{P}(\sqrt{\varepsilon}\xi \approx \rho) \approx \exp\left(-\frac{1}{2}\varepsilon^{-1} \int_{\mathbb{R}^2} \rho^2\right).$$

This is an important ingredient for us to study the LDP of stochastic PDE.

Thank you!